AN INTRODUCTION TO THE THEORY OF ELECTRICITY.

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TO THE

THEORY OF ELECTRICITY

WITH NUMEROUS EXAMPLES

BY

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PREFACE TO THE FOURTH EDITION.

THE demand for another edition of the present work first issued in 1876 is very gratifying to the author.

The chief additions in the present edition consist of articles upon the Magnetic Circuit and on the Dynamo.

Having worked through the book in recent years with his pupils the author has learnt how much inaccuracy, obscurity and inadequacy of explanation remained even in the last issue, and he has spared no pains to remove such blemishes in the present edition.

He also takes the opportunity of thanking numerous friends both at home and in America for sending him frequent communications concerning one part and another of the book, all of which have received careful consideration.

His thanks are also specially due to his former pupil, Mr E. H. Cozens Hardy of the City Guilds Institute, who has read through nearly the whole of the proofs and made many valuable suggestions and corrections.

L. CUMMING.

RUGBY,
October 1, 1894.

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CHAPTER I.

PHYSICAL UNITS.

1. THE measurement of all physical quantities depends ultimately on the units of *space*, *time*, and *mass*.

In England the units are generally the *foot*, *second*, and *pound*; but we shall adopt the centimetre, gramme, and second (C. G. S.) system, which is now almost universally used for scientific purposes.

2. DEF. VELOCITY *is the rate of motion of a body, and, if uniform, is measured by the number of centimetres passed over per second; if variable, it is measured, at any instant, by the number of centimetres which would be passed over per second supposing the velocity uniform during that second, and of the same value as at the instant under consideration.*

It will be seen from the above definition, that velocity is a property of a body at any given instant, and not necessarily the same during a finite interval. Thus when we speak of a train as going thirty miles an hour, we do not mean to say that it has gone thirty miles in the past hour, or that it will go thirty miles in the next hour; but that supposing the velocity to remain uniform, it would go thirty miles during that time.

The unit of velocity is the velocity of a body which goes over one centimetre per second.

If a body moving uniformly with velocity v pass over a space s in time t, the relation between these quantities is clearly expressed by the formula

$$s = vt \quad \text{.............................(i).}$$

3. If two motions are impressed on a body, as when a person walks across a carriage in motion, it is easy to see that the motions may be compounded by the parallelogram law. For if one motion carries a point from A to B and another simultaneously from A to C, then if both movements take place the point is transferred from A to D, AD being the diagonal of a parallelogram, having AB, AC for adjacent sides.

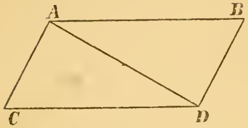

Fig. 1.

If both movements took place with uniform velocity in one second, then AB, AC would represent the numerical values of the two velocities, and geometry shows that under these velocities combined, the body moves uniformly from A to D in one second. Then AD represents the resultant velocity of the two components AB, AC.

4. Def. Acceleration *is the rate of change of velocity, and is measured, when uniform, by the number of units of velocity added on to a body's motion per second. When variable, it is measured at any instant by the number of units of velocity which would be added on per second, supposing the acceleration constant, and of the same value as at the instant under consideration.*

Like velocity, acceleration is a property of a body at a particular instant, not necessarily continuing the same through a finite interval. It is the measure of the body's *quickening* at that instant.

The unit of acceleration is the acceleration of a body whose velocity increases by a unit of velocity per second. If a body be moving under a uniform acceleration f through a time t, and if V be the initial velocity, and v the velocity at the end of the time t, then

$$v = V + ft \qquad \text{(ii).}$$

The best illustration of a uniform acceleration is the motion of a body near the earth's surface. In this case it is proved by experiment that the acceleration due to the earth at the sea level is represented numerically by 981 (at

Paris). Thus a body falling to the earth has its velocity increased each second by 981 centimetres per second. This does not mean that the body describes 981 centimetres in the second, or even describes 981 centimetres less in one second than in the next, but that if, for instance, the body is projected downwards with a velocity of 100 centimetres per second, it will have at the end of the first second a velocity of 1081 centimetres per second, at the end of the second second its velocity will be 2062 centimetres per second, and so on during each second of the motion.

Retardation is treated as negative acceleration. If for instance a body be projected upwards, its velocity is diminished by 981 cm. per second each second, and generally if f represent the retardation, our formula (ii) becomes

$$v = V - ft \quad \text{(ii}'\text{)}.$$

If the resulting velocity should be negative it will denote that the body is moving with a certain velocity in the direction opposite to that of projection.

5. To find the space described during a given time t by a body moving with uniform acceleration, we may consider that since the acceleration is uniform, the *average velocity* during the interval will be the same as the velocity at the middle of the interval, and this will clearly be $V + \frac{1}{2}ft$. The space described will be the same as that due to this velocity during the time t. Hence, by formula (i),

$$\begin{aligned} s &= (V + \tfrac{1}{2}ft)\,t \\ &= Vt + \tfrac{1}{2}ft^2 \quad \text{(iii)}, \end{aligned}$$

or if the acceleration is negative,

$$s = Vt - \tfrac{1}{2}ft^2 \quad \text{(iii}'\text{)}.$$

Combining (ii) and (iii) by algebra, we have from (ii)

$$v = V + ft;$$

or

$$\begin{aligned} v^2 &= V^2 + 2Vft + f^2t^2 \\ &= V^2 + 2f(Vt + \tfrac{1}{2}ft^2), \end{aligned}$$

from (iii)

$$= V^2 + 2fs \quad \text{(iv)}$$

Similarly, from (ii′) and (iii′),

$$v^2 = V^2 - 2fs \quad\text{.....................(iv′).}$$

6. The space described may also be illustrated graphically by a method which will be of frequent use.

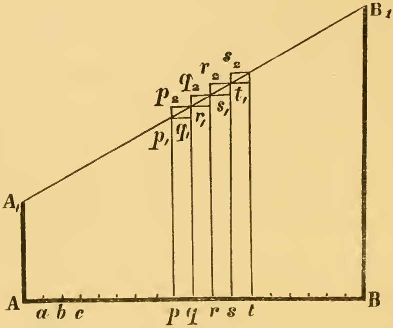

Fig. 2.

Set off along a horizontal line AB equal lengths Aa, ab, bc, &c., representing short intervals into which the whole time AB of the motion of the body can be divided, raise at A, a, b, &c. straight lines (called ordinates) perpendicular to AB, and of such lengths as to represent on a certain scale the velocities of the body at the beginning and end of each interval. Let these lines be AA_1, aa_1, $bb_1 \ldots pp_1$, qq_1, $rr_1 \ldots BB_1$. Through p_1, q_1, r_1 etc., draw lines parallel to AB, to meet pp_1 produced in p_2, qq_1 in q_2, etc., and produce them to complete the system of small parallelograms. The space described during the small interval of time pq will be represented numerically by something between $pq \times pp_1$ and $pq \times qq_1$, or by some area between p_1q and p_2q; since pp_1 represents the velocity at the beginning of the interval and qq_1 the velocity at the end. The whole space described will be intermediate between the sum of all the parallelograms p_2q, q_2r, r_2s, &c., and the sum of all the parallelograms p_1q, q_1r, r_1s, &c. The difference of these two sums is clearly a parallelogram whose height is $(BB_1 - AA_1)$ and base one of the intervals pq, and therefore equals $(BB_1 - AA_1) \times pq$, and if the

intervals are sufficiently small this difference is indefinitely small, and each of the sums becomes the same as the whole area A_1ABB_1, and this therefore will represent the whole space described. Since the acceleration is uniform, the increments of velocity are the same for the same increments of time, and consequently by Euc. VI. Prop. ii. the line A_1B_1 is a straight line, also $AA_1 = V$, and $BB_1 = V + ft$, and $AB = t$. Hence the area of the trapezium

$$\begin{aligned} &= \tfrac{1}{2}(AA_1 + BB_1)\,AB \\ &= \tfrac{1}{2}(V + V + ft)\,t \\ &= Vt + \tfrac{1}{2}ft^2, \end{aligned}$$

which agrees with our formula (iii).

It must be carefully noted that the area A_1ABB_1 has no actual relation to the space described beyond the numerical one here represented. Thus if seconds be represented by centimetres along AB, and units of velocity by centimetres perpendicular to AB, then the number of square centimetres enclosed by the lines AA_1, BB_1, AB, A_1B_1 represents numerically the number of units of space passed over by the body in the time AB.

When in future we make use of this graphical representation of a formula, we shall indicate its construction by saying that *abscissæ*, or distances set off along a horizontal line such as AB, are to be taken to represent the number of units in one magnitude, and *ordinates* or lines perpendicular, to represent some other co-related magnitude; and from the nature of the figure so formed, we shall deduce by geometry various relations.

7. In dealing with force we must consider the material bodies through which all forces are made sensible to us. Although matter is as hard to define as space and time, we need a means of measuring it. Its measure is called mass, the unit being a certain lump of matter called the standard mass (or by the general public the standard weight). This standard in Britain is the pound Avoirdupois, but for scientific purposes we use the gramme, which by definition is the mass of a c.cm. of water at 4° C.

Other masses are compared with the gramme by weighing, that is, finding the ratio of the pull of the earth's gravitation on the body to the pull on the gramme at the same time and place. This ratio is called the mass of the body.

Experience shows that bodies weigh very differently according to the material of which they are made. Thus a mass of iron weighs much more than a mass of wood of the same volume, but far less than a mass of gold also of the same volume. This difference is expressed in two ways. (1) We may weigh a certain volume of each body and record the mass (or as people say the weight). If the volume be a unit of volume the mass is called the density of the material. Thus a c.cm. of copper will be found to weigh 8·9 grammes, and this is expressed by saying that the density of copper is 8·9 grammes to the c.cm. The same might be expressed in lbs. to the cu. ft. or in any other measure we choose. (2) We may take some standard substance, that usually chosen being water, and compare the weight of any volume of the body with the weight of the same volume of the standard substance. This ratio is called the specific gravity of the body, and is independent of any system of measurement.

Since the gramme is the mass of a c.cm. of water, the number of grammes to the c.cm. of any other substance represents also its specific gravity. Thus the specific gravity of copper is 8·9.

The following are the formal definitions.

DEF. MASS *of a body is the ratio of the gravitation pull on the body to that on the standard mass.*

DEF. DENSITY *of a substance is the number of units of mass in one unit of volume of the given substance.*

DEF. SPECIFIC GRAVITY *of a substance is the ratio of the weight (or mass) of a volume of that substance to the weight (or mass) of the same volume of a standard substance.*

From the definition of density it follows that if M, V, D represent the mass, volume and density of a body $D = \frac{M}{V}$ or

$$M = DV \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

When we speak of a particle of matter, we mean a mass of matter which can be acted on by forces, but which in its geometrical relations can be treated as a point.

8. Def. Momentum or Quantity of Motion *is defined by the product of the mass of a body into the velocity with which it is moving.*

Unit of Momentum is the momentum of unit mass moving with unit velocity.

Since the momentum of a given body is proportional to its velocity we can compound momenta by the parallelogram law, as explained above for velocities.

9. Def. Force *is defined as that which changes or tends to change a body's state of rest or motion, and any given Force may be measured by the change of motion it produces per unit time.*

Thus a unit force is that which imparts to a gramme a unit acceleration, and is called a *dyne.*

10. The science of Physics is founded upon certain experimental truths, which were first given concisely by Newton. They are called the Laws of Motion, and we will give them simply translating Newton's own words, and adding explanations chiefly derived from his own scholia or comments on these laws.

Law I. *Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by impressed forces to change that state.*

This is often called the law of Inertia of Matter, expressing that matter has no tendency to move without the application of force. It is impossible to establish it experimentally, as every body in the universe is moving, and subject to a great complexity of forces. We may, however, establish *relative rest,* as of a body resting on a horizontal plane, and we moreover observe not only that it never sets itself in motion, but that when the body is started the smoother the plane the more slowly is the velocity diminished.

This law indicates the convention universally adopted for the measurement of time, namely by the motion through equal spaces of a body acted on by no external forces. This condition is most nearly fulfilled by the earth in respect of its rotation on its own axis, when the angular motion is uniform, and is always taken as the practical means of measuring time.

11. LAW II. *Change of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

Newton in this law says nothing about the state of rest or motion of the body on which the force acts. Hence if the body is already moving in the direction of the force's action the change in motion is simply added to the already existing motion. If the motion be not in the same direction as the force the change in motion is compounded with the already existing motion according to the parallelogram law.

This is shown experimentally by dropping a stone from the mast-head of a ship which is moving uniformly. The stone is found to fall at the foot of the mast, both in the same time, and in the same position, as if the ship had been at rest; the pull of gravity on the stone not being interfered with by its own uniform horizontal motion, which was necessarily that of the ship at the moment it was dropped.

12. Newton again says nothing about there being only one force acting, and we conclude that if several forces are acting we may compound the change of motion due to each of them just as we have shown above we can compound motions themselves by the Parallelogram Law.

Thus if AB, AC measure the momenta which two forces acting in the directions AB, AC would if they acted separately impress on the body: the resultant momentum will be represented by the diagonal AD of the parallelogram of which AB, AC are adjacent sides.

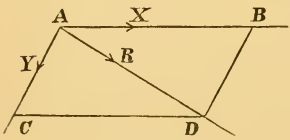

Fig. 3.

This shows us that velocities, accelerations, momenta and forces can all be compounded according to the Parallelogram Law.

13. The second law gives the proper measure of force. For, put into modern language, it says that force is measured by the change in motion (i.e. momentum) it produces in unit time. Since mass is constant, the change in momentum is the product of the mass into the change of velocity, and change in velocity per unit time is simply (Art. 4) acceleration. Thus the proper measure of force is the product of mass and acceleration. Thus if P be a force which causes acceleration f in mass m we have

$$P = mf.$$

14. The converse of compounding physical quantities is called resolution. Thus if AD in Fig. 3 represent any velocity, momentum or force whose measure is R, and we require to replace it by two quantities of the same kind acting along AB, AC, we can clearly do so by completing the parallelogram $ABDC$, when AB and AC will represent two quantities which when compounded together produce the resultant AD. Then the measures of AC, AB are called the components of R in these directions. The magnitudes of these components will by the figure be given by

$$X\text{, the component along } AB = \frac{AB}{AD} R,$$

$$Y\text{,} \quad \text{,,} \qquad \text{,,} \qquad \text{,,} \quad AC = \frac{AC}{AD} R.$$

If AB, AC be at right angles to each other, and α the angle between AD and AB, then

$$X = R \cos \alpha, \quad Y = R \sin \alpha, \quad \frac{Y}{X} = \tan \alpha \ldots\ldots \text{(vi)}.$$

In Fig. 3, if the direction of R be reversed, the three components X, Y, $-R$ form a system in equilibrium, each being equal and opposite to the resultant of the other two. The directions of X, Y, $-R$ will be parallel to the sides of

the triangle ABD taken in order, X acting in direction AB, Y in direction BD, and R in direction DA. Hence it follows that if there be acting on a particle three forces which are parallel in direction, and proportional in magnitude, to the sides of any given triangle, they shall be in equilibrium. This proposition is known as the triangle of forces. And it is obvious that there is a corresponding triangle of velocities, accelerations, and momenta.

15. Of the nature of Force are all weights, pressures, tensions of strings, attractions and repulsions between bodies.

It will be convenient to express the unit Force in terms of our standard weight. That taken as the ordinary standard is the weight of a gramme at the sea-level in the latitude of Paris. Now it is known by experiment that in this latitude the acceleration of a falling body is 981. Hence the unit of weight is a gramme under 981 units of acceleration. Therefore a unit of Force $= \frac{1}{981}$ of the weight of a gramme, and is called a dyne.

In future, weights will be measured in grammes and converted into absolute units of force by multiplication by 981. If a weight be given as w grammes the measure of it in dynes is $981w$.

16. Law III. *To every action there is an equal and opposite reaction; or the mutual actions of any two bodies are always equal and oppositely directed.*

This expresses the fact that when a body is pressed, it presses back with an equal force.

If for example I press my finger on the table, the table presses my finger back with the same force with which I press the table. If a horse tow a boat along a canal the horse is dragged back with exactly the force it uses to drag the boat forward.

This law shows that the forces between two bodies, or parts of the same body, consist of pairs of equal forces acting along straight lines in opposite directions. Taking such a

pair of forces F on a body A and $-F$ on a body B; which may either be action and reaction along the line joining the bodies, or these resolved in any given direction; then F and $-F$ measure at any instant the rates of change of Momentum of the respective bodies along this direction. The quantities of momentum imparted to the two bodies during any short interval t will be Ft and $-Ft$, so that if the momenta at first along this line were M_A, M_B, after the time t during which this force F has been in action the momenta will be

$$M_A + Ft \text{ and } M_B - Ft,$$

and therefore the total momentum of the two bodies in this direction remains unchanged. And since what is true for each such interval will be true for the sum of any number of such intervals, it follows that during any finite time the sum of the momenta of two bodies in any given direction is constant, the effect of action and reaction being an exchange of momentum between the bodies and not absolute loss or gain.

This principle is applicable to the case of impact or friction between two free bodies, but we must be careful not to apply it in the case of impact or friction against the Earth, since we are unable to measure the change in the Earth's momentum due to the friction or impact.

17. Newton adds to the Third Law of Motion this important scholium.

"*If the Action of an agent be measured by its amount and velocity conjointly: and if similarly, the Reaction of the resistance be measured by the amounts of its several parts and their several velocities conjointly whether they arise from friction, cohesion, weight or acceleration; Action and Reaction, in all combinations of machines, will be equal and opposite.*"

Newton conceives the system of bodies (or combination of machines) in motion, the point of application of each force having a certain velocity. If this velocity of the point of application be resolved along the direction of the force its component (as he explains elsewhere) is the velocity of the force: and the measure of this velocity multiplied into the

measure of the force gives the "Action." This action is simply the rate at which the force works, for which Watt afterwards invented the unit of the horse-power, which represents the rate of work of an agent which raises 33000 lbs. against gravity one foot per minute, or in other words which moves 33000 lbs. against gravity with the velocity of one foot per minute. Newton's statement therefore tells us that when a system of forces acts on a material system without causing acceleration (i.e. when the system is in equilibrium) the sum of the rates at which all the forces are working must be zero. In computing the rates those must be counted as negative in which the velocity of the force is in an opposite direction to the force, as when a weight is lifted against gravity. This is the well-known principle in Mechanics that "what is gained in power is lost in time."

18. Next we notice that velocities are proportional to the spaces described during the same very short interval. If therefore a small displacement be made consistently with the geometrical relations, the velocities of the points of application are simply proportional to their respective displacements, and the resolved parts of the displacements will be proportional to the velocities of the forces. The product of the force into this resolved displacement is obviously the work done during the displacement, counted positive when the displacement is concurrent with the force and negative when opposed to it. The products are what are called in Mechanics the Virtual Moments of the Forces, and Newton's principle is the Principle of Virtual Velocities.

If all the forces remain in the same direction during a finite displacement, we may still apply the principle which then becomes the "principle of work" in Mechanics.

19. Again, if the small displacement be a rotation about a given line and the force acts in the plane of the displacement, we may assume its point of application shifted to the foot of the perpendicular on its line of action drawn from the axis. All points in the system will be displaced through the same small angle, θ, suppose, and if p be the length of the perpendicular on the force's line of action, $p\theta$ will be the small

displacement whose direction coincides with the line of action. Hence the "action" of the force will be proportional to $\pm Fp\theta$ where F measures the force, taking the + sign when the force is in the direction of the displacement, and the − sign when in the opposite direction. Thus if we have a series of forces $F_1, F_2 \ldots$ and $p_1, p_2 \ldots$ be the perpendiculars on their lines of action, Newton's principle gives us

$$F_1 p_1\theta + F_2 p_2\theta + \ldots\ldots = 0,$$

$$\therefore\ F_1 p_1 \ + F_2 p_2 \ + \ldots\ldots = 0, \ldots\ldots\ldots\ldots \text{(vii)}$$

which is the principle of moments in Mechanics.

20. Def. Work *may be defined as resistance overcome through space.*

No work is done in moving a body perpendicular to the Resultant force acting on it. A body under gravity only could be moved horizontally forward without the exercise of any force, and if once started would move on a smooth level surface uniformly for ever. In computing the work done during any given movement we must take account only of the component of the force along the direction of motion of its point of application.

Thus if a body be carried from one position to another against a force, such as a weight raised, a certain amount of muscular or other power must be expended. This work is measured, when the acceleration is uniform, by the mass moved into the acceleration against which it is moved into the space through which it is moved. If W be the work expended in moving a mass m, against an acceleration f, through a space s, we have

$$W = mfs \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{(viii)}.$$

The unit of work will be done in moving a gramme through a centimetre against a unit of acceleration, and is called an *erg*. The ordinary English unit of work is the foot-pound, being the work done in raising a pound through a foot.

Rate of work is the number of ergs done per second by any machine.

The English standard of working power is the "horse-power," defined as 33000 foot-pounds per minute.

To convert this into ergs per sec. we have only to remember that

$$1 \text{ centimetre} = \cdot 3937079 \text{ inch},$$

$$1 \text{ gramme} = 15 \cdot 43235 \text{ grains}.$$

Hence $$1 \text{ foot} = \frac{12}{\cdot 3937079} \text{ cms.},$$

$$1 \text{ lb. Avoirdupois} = 7000 \text{ grains Troy},$$

$$= \frac{7000}{15 \cdot 43235} \text{ gms.}$$

Therefore $$1 \text{ foot-pound} = \frac{12 \times 7000}{15 \cdot 43235 \times \cdot 3937079} \text{ cm.-gms.},$$

which, remembering that gravity = 981 units of acceleration, becomes

$$= \frac{12 \times 7000 \times 981}{15 \cdot 43235 \times \cdot 3937079} \text{ ergs}.$$

Therefore one horse-power

$$= 33000 \text{ foot-pounds per } 1',$$

$$= \frac{33000}{60} \text{ foot-pounds per } 1'',$$

$$= \frac{33000 \times 12 \times 7000 \times 981}{60 \times 15 \cdot 43235 \times \cdot 3937079} \text{ ergs per } 1'',$$

$$= 7460 \text{ million ergs per } 1'' \text{ nearly}.$$

21. DEF. MOMENT OF A FORCE. *The moment of a force about a given point is defined as the product of the force into the perpendicular from the given point on to the line of action of the force.*

The moment of a force thus defined measures the tendency of a force to turn a body about a given point as axis. And it is obvious that when the forces on a body balance each other, the sum of the tendencies of all forces which

twist it in one direction must exactly balance the tendencies of all forces twisting it in the opposite direction (Art. 19).

22. Def. Couple. *Two forces which are equal in magnitude and parallel, acting in opposite directions, but not in the same straight line, are termed a couple.*

It is clear that, if we take any point O (Fig. 4) in the plane of the forces, and from it draw a perpendicular Oab to the two forces, the difference between their moments about O is always $P \,.\, Ob - P \,.\, Oa = P \,.\, ab$. Thus either force multiplied by the perpendicular distance between the forces is called the *moment of the couple*, and measures the tendency of the couple to twist the body round.

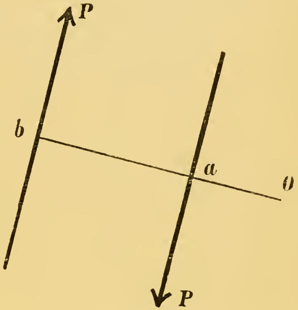

Fig. 4.

From this it follows, that no single force can balance a couple. For if possible suppose it to be balanced by any force. Then choosing O a point in the line of the force, its moment about O vanishes, and there remains the moment $P \,.\, ab$ unbalanced (Art. 19).

23. Remembering that the action of an agent is proportional to the work done by the agent during a given displacement, Newton's principle shows that work done on a system of bodies has its equivalent in work done against gravity as in raising a weight, against molecular forces as in compressing a spring or against friction, if there be no acceleration: but if there be acceleration some of the work done may be done against the resistance of the body to acceleration. To measure this latter part, let us assume that a particle of mass m when displaced through a space s receives an acceleration whose measure is f, the displacement being assumed resolved in the direction of f. Then the resistance must be measured by mfs. Now if V be the velocity at the beginning and v at the end of the space s,

resolved along the direction of the acceleration, we have shown (Art. 5)

$$v^2 = V^2 + 2fs,$$

$$\therefore\ \tfrac{1}{2}mv^2 = \tfrac{1}{2}mV^2 + mfs,$$

$$\therefore\ mfs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2.$$

If the motion be not in the direction of the acceleration the velocity will have at each end a component u at right angles to the direction of motion. The total velocities will then be $\sqrt{V^2 + u^2}$ at first and $\sqrt{v^2 + u^2}$ at last and

$$\tfrac{1}{2}m\,(v^2 + u^2) - \tfrac{1}{2}m\,(V^2 + u^2) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mV^2 = mfs.$$

This proves that if we compute at each successive point in the body's motion the product of one half the mass into the square of velocity, the work done against acceleration from point to point is measured by the change in this product. The name given to this physical quantity is Kinetic Energy.

24. DEF. KINETIC ENERGY *is defined as half the product of the mass into the square of the velocity of a body.*

If a body of mass m be raised against gravity to a height h an amount of work whose measure is mgh has been expended on the body. If the body be allowed to fall freely back again it will on returning to its initial position have an amount of kinetic energy which the equation $\tfrac{1}{2}mv^2 = mgh$ proves to be numerically equal to the work expended in raising the body. The body in the higher position had therefore in virtue of the work expended upon it an advantage in respect of energy over the same body at the lower level, in that it had a capacity for acquiring kinetic energy by simply allowing gravity to act upon it. This kind of energy is called *Potential Energy.*

25. Taking the case of a ball projected upwards with velocity V. If the velocity be v when the ball has risen through a space s, we have by Art. 23,

$$\tfrac{1}{2}mV^2 - \tfrac{1}{2}mv^2 = mgs;$$

or

$$\tfrac{1}{2}mV^2 = \tfrac{1}{2}mv^2 + mgs.$$

Here we see that at the moment of projection the body had kinetic energy measured by $\frac{1}{2}mV^2$, and that throughout the whole subsequent motion the kinetic energy ($\frac{1}{2}mv^2$) is less than that at the beginning of the motion by mgs or the weight of the body multiplied by the space through which it is raised; but this is the work done in raising the body through space s, and is the measure of the potential energy of the body. Hence the above equation is a statement of the fact that the sum of the Potential and Kinetic Energy of the body remains the same during the whole motion. As the body rises it is losing kinetic and gaining potential energy, when it has reached its highest point its whole energy is converted into potential energy, when it begins to descend again, it gains during descent, kinetic at the expense of its potential energy, until on its return to the point of projection it has the same kinetic energy as at starting, though its velocity is in the opposite direction.

26. In the case of a pendulum the same principle holds good. The original impact by which the pendulum is started communicates to it kinetic energy, and the bob comes to rest when the potential energy due to its rise is equivalent to the kinetic energy imparted.

Thus, if a pendulum bob of mass m be started with a velocity V at A, and if on reaching B the velocity is destroyed, then, if through B a line BN be drawn horizontally, the bob has been lifted through AN.

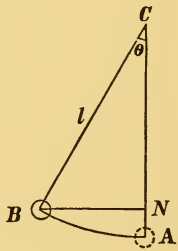

Fig. 5.

$$\therefore\ mgAN = \tfrac{1}{2}mV^2,$$

where g denotes the acceleration due to the attraction of the earth on the bob.

But $$AN = l(1 - \cos\theta)$$
$$= 2l\sin^2\frac{\theta}{2},$$

where $$l = \text{length of pendulum};$$

$$\therefore\ mg2l\sin^2\frac{\theta}{2} = \tfrac{1}{2}mV^2;$$

$$\therefore \quad V^2 = 4lg \sin^2 \frac{\theta}{2};$$

$$V = \sin \frac{\theta}{2} \sqrt{4lg},$$

$$mV = m\sqrt{4lg} \sin \frac{\theta}{2}.$$

Hence with a given pendulum the momentum at starting is proportional to the sine of $\frac{1}{4}$ of the whole angle of swing, or to the chord of half the arc of swing.

But mV measures the blow by which the pendulum was started. Hence in any case of an instantaneous force applied to a pendulum, we shall assume the blow is proportional to the sine of half the angle of deflection. This is called the principle of the ballistic pendulum.

The same principle will be applied further on to a magnet swinging in a uniform field.

27. Other instances of Potential energy are a compressed spring where work has been done against molecular forces in compression, such that on rebound the spring acquires the same amount of energy of the kinetic kind, as was expended in compressing it. When work is expended in heating a body it was considered by Newton and long after his time that Work was absolutely lost, as in the case of friction. Dr Joule's experiments have now shown that there is a strict equivalence between the amount of work expended and the amount of heat gained, and that where heat is used to do mechanical work there is the same equivalence between the heat put out of existence and the mechanical work gained. Thus we are now compelled to regard heat as itself a form of energy obeying the same laws as other kinds of energy. As it is probable that Heat is to be regarded as a kind of motion amongst the molecules of the body it is a variety of Kinetic energy. In chemical dissociation we have also an instance of Potential energy as in the elements of gunpowder, or dynamite, or in the fuel of a fire and the oxygen of the air, in all of which a large amount of energy may be obtained by an expenditure of an indefinitely small amount as in loosening

the detent of the trigger in a gun or putting a match to the fuel in the fire. Electrical separations and electrical currents we shall afterwards see to be also illustrations of potential energy obeying the same laws.

28. DEF. ENERGY. *Energy is defined to be capacity for doing work, and may be either* (1) *Kinetic, when the body is in absolute motion, or* (2) *Potential, when the body in virtue of work done upon it, has acquired a position of advantage, so that work can at any time be recovered from it, by the return of the body to its old position.*

Taking these views we now state Newton's scholium:—Where work is done on a system of bodies it has its equivalent either in Kinetic energy of the system or in work done either (1) against gravity as in raising a weight, (2) against molecular forces as in compressing a spring, performing a chemical dissociation, making an electrical separation or an electrical current, (3) against friction in which we obtain an equivalent amount of heat—itself a form of Kinetic energy.

If no work is done on the system, there can be no gain or loss of energy in the system, the forces between different parts of the system merely developing Kinetic at the expense of Potential energy or vice versa.

Thus in any material system acted on by no external forces we have two absolute constants, its amount of matter and its amount of energy, neither of which can be increased or diminished. The matter of the system may by chemical action be changed into other kinds of matter, but always of the same total amount, and the energy may be converted into all forms of which energy is susceptible, but also without altering the total amount.

This is the principle of Conservation of Energy of which the foundation was laid by Newton; but his remark was passed by unnoticed till attention was called to it in Thomson and Tait's *Treatise on Natural Philosophy* (published in 1867), to which the reader is referred for a fuller treatment of the subject.

EXAMPLES ON CHAP. I.

The following relations may be assumed:

1 metre = 39·3708 inches.
1 pound avoirdupois = 453·59 grammes.
1 cubic foot of water weighs 1000 oz. Av.

The acceleration of gravity = 32·2 when ft. and sec. are fundamental units.

The abbreviation cm. is used for centimetre.
.... gm. .... gramme.
.... sec. .... second.
.... sq. .... square.
.... cu. .... cubic.
.... den. .... density.

1. Express in metrical units the velocity of sound which travels 1100 feet per sec. *Ans.* 33527·4 cm. per sec.

2. How many yards per minute and miles per hour are described by a body which travels at the rate of 1000 cm. per sec.? *Ans.* 656·18; 22·37.

3. The acceleration of gravity is measured by 981 in the metrical system. Find its numerical value when feet and seconds are employed as fundamental units. *Ans.* 32·2.

4. Show that in British measure (ft., lb., sec. being fundamental units) the absolute unit of force (called the poundal) is very nearly the weight of half an ounce Avoirdupois.

5. Find the density in lbs. to the cu. ft. and the specific gravity of a stone of which 10·5 cu. ft. weigh three-quarters of a ton. *Ans.* 160; 2·56.

6. A cannon-ball is shot vertically upwards and ascends for five seconds, then returning back again,

(i) With what velocity was it projected?

(ii) What height did it reach?

(iii) What time elapses between leaving the gun and returning to earth?

(iv) If it was caught at the instant of turning and hurled down with a velocity of 1000 feet per second, what would be its velocity on reaching the ground?

(v) In the last case how long would it take during its fall? *Ans.* $\frac{64}{161}''$ nearly.

7. The Moon's distance is 60 times the Earth's radius. Through what distance does the Earth pull the Moon every minute? Assuming that the Moon moves in a circle, and that the radius of the Earth is 4000 miles, calculate the length of the lunar month. *Ans.* 16·1 ft.; 27·4 days.

8. A person drops a stone down a well, and hears the splash after 2·86 sec. Find the depth of the well, making allowance for the time taken by the sound in coming up (see Ex. 1). *Ans.* 121 feet.

9. A balloon ascends vertically and uniformly for 4·5 sec., and a stone is then let fall which takes seven sec. to reach the ground. Find the velocity of the balloon and its height when the stone was dropped.
Ans. 68 ft. per sec.; 306 ft.

10. A stone after falling for one sec. strikes a pane of glass and loses half its velocity. How far will it fall in the next second? *Ans.* 32 ft.

11. What is the volume in c.cms. of 1·5 kilogms. of iron whose density is 7·25 gms. per c.cm.? *Ans.* 206·9.

12. What is the weight in grammes of 10,000 c.cms. of sea-water whose density is 1·028 gms. per c.cm.?

13. Two forces whose magnitudes are in the ratio of 3 to 4 act at right angles to each other; what is the magnitude of their resultant?

14. Two equal forces have a resultant also equal to either of them; at what angle are the two components acting?

15. It is required to substitute for a given vertical force two forces, one horizontal and the other inclined at an angle of 45° to the vertical. Find the ratio of the two components to the original force.

16. If a body is falling down an inclined plane, show how to compute the part of gravity which is acting upon it in the direction of motion.

17. A body is falling down an inclined plane without friction, the angle of elevation of the plane being 30°. Find the space it will pass over in the two first seconds from rest.

18. Two weights are attached to the ends of a string without weight, and are slung over a smooth pulley. Give an expression for the acceleration acting on the system if it is free to move.

19. If the weights in the preceding question be 20 and 10 gms., how many cms. will the larger weight have fallen from rest at the end of 3 secs.? *Ans.* 1471·5.

20. Illustrate the meaning of the term work by giving a list of examples of cases in which work is done *on* a body, and also a list of cases in which work is done *by* a body.

21. Discuss the principle of conservation of energy as applied (i) in the case of two bodies moving with mutual friction; (ii) in the case of impact between two bodies.

22. Show how the sun's energy is employed to grind corn, (i) by means of a wind-mill; (ii) by means of a water-mill.

23. Two balls M, M', moving in the same line with velocities V and V', impinge. Show that there will be a loss of energy during the impact unless the whole momentum exchanged be equal to $\frac{2MM'}{M+M'}(V-V')$.

24. Show that the energy stored up in a reservoir of water standing on the ground is measured by half the product of its depth into its weight.

25. Show that if an additional quantity of water is to be added to the reservoir, it will be immaterial whether it is raised up and poured in from above or forced in at the bottom.

26. Show also from the principle of conservation of energy, that if an orifice be made in the bottom of the reservoir and the water escape without friction, the velocity of the issuing stream will be that due to a fall under gravity from a height equal to the depth of the water.

27. In ques. 26 what would be the velocity if the vessel were filled with mercury whose density is 13 times that of water?

28. Prove that the work done in lifting a body up an inclined plane is equal to the work done in lifting it vertically through the height of the plane.

29. Compare the momentum of a cannon-ball of 600 lbs. moving at the rate of 1000 feet per second, with that of an express train of 100 tons moving at the rate of 40 miles per hour. *Ans.* Ratio of 225 to 4928.

30. Compare also the work done in stopping the cannon-ball and train in the preceding question.

Ans. Ratio of 27 to 34·7 nearly.

31. A block of wood weighing one cwt. less 4 oz. is suspended by a string, and is struck horizontally by a bullet weighing 4 oz., which sinks into the block and causes it to ascend six inches. Calculate the velocity of the bullet in feet per second. *Ans.* $1792\sqrt{2}$.

32. A bullet moving at the rate of 1000 feet per second penetrates three inches into a fixed block of wood. Calculate the velocity necessary to cause it to penetrate eight inches.

33. Compare the masses of two cylindrical bullets which proceeding from cannons of the same bore, with the same velocity, penetrate 8 and 12 inches respectively into the same block of timber.

34. A bullet weighing 240 grammes, and moving with a velocity of 300 metres per second, penetrates 4 cms. into a block of timber, the area of section being 8 sq. cms. Compare the retardation of the timber per sq. cm. with the acceleration of gravity. *Ans.* 14335 to 1 nearly.

35. Find the kinetic energy of a ring which makes a given number of revolutions per second round an axis passing through its centre and perpendicular to its plane.

Let a be the radius and m the mass of the ring: also let it make n revolutions per second.

Each particle of ring describes $2\pi an$ cm. per sec.;

$$\therefore \text{ velocity of each particle} = 2\pi na,$$

$$\text{and mass of ring} = m.$$

$$\text{Hence kinetic energy} = \tfrac{1}{2} m \,.\, (2\pi na)^2$$

$$= 2\pi^2 n^2 m a^2.$$

36. Find the kinetic energy of a solid disc revolving about an axis through its centre making n revolutions per second.

The disc may be regarded as made up of a series of narrow rings. If O be the centre, and OPQ be drawn through one of the rings, the mass of the ring $= 2\pi\rho . OP . PQ$, where ρ is the mass per unit of area.

$$\begin{aligned} \text{Hence energy of ring} &= 2\pi^2 n^2 OP^2 . (2\pi\rho . OP . PQ) \\ &= 4\pi^3 \rho n^2 . OP^3 . PQ \\ &= \pi^3 \rho n^2 . 4OP^3 (OQ - OP) \\ &= \pi^3 \rho n^2 (OQ^4 - OP^4), \text{ see Art. 38,} \end{aligned}$$

whence adding all the successive elements,

$$\text{kinetic energy of disc} = \pi^3 \rho n^2 a^4 = \text{mass} \times \pi^2 n^2 a^2.$$

CHAPTER II.

THEORY OF POTENTIAL.

29. OUR knowledge of physics is a knowledge only of the forces exerted by matter under a variety of conditions. Near any material system such as the Earth we find that if we try to move a mass of matter from one position to another the movement is either resisted, and work has to be done in moving the mass, or if we move it in the opposite direction a force assists the movement and would, if the mass were allowed to fall by frictionless constraint, generate in it energy during the fall. To express this condition in any space we use the term *Field of Force.*

DEF. FIELD OF FORCE *is any bounded or unbounded region in which any two points A, B being taken, work has to be done to move a mass of matter from A to B, while kinetic energy is generated if the mass be allowed to fall without friction from B to A.*

We shall assume that the system of force is a Conservative System, so that the work done in carrying the matter from A to B is numerically equal to the Kinetic energy acquired by it in falling from B to A.

30. At any point in a field of force there exists a certain definite direction of the resultant force on a particle at that point: this direction is the line along which the particle, if left to itself, will tend to fall.

By choosing points near enough together, so that the line joining each two consecutive points shows the direction of the resultant force on a particle at a point on that line, we shall have a broken line through the field such that its direction at every point shows the direction of the resultant force near

that point. If the points be taken close enough together, this broken line becomes a continuous curved line, such that the tangent at every point shows the direction of the resultant force on a particle at that point. This line is then a Line of Force.

DEF. LINE OF FORCE *is a line in a field of force such that the tangent to the line at any point shows the direction of the resultant force on a particle at that point in the field.*

It is clear that one line of force passes through every point in the field, and that lines of force cannot intersect, since if they could there would be at their intersection two directions of the resultant force, an obvious impossibility.

31. The magnitude of the force by which a mass of matter at a point is urged along the line of force, depends (Art. 16) jointly on the field or system of force and on the quantity of matter. If we wish to express the variation in force at different points in the field, we must choose some definite quantity of matter as testing unit, and find the force it experiences when placed at the different points. The most natural quantity to choose is of course the *gramme* or unit of mass, and the term Strength of Field at a point is employed to express the force experienced by a gramme when placed at that point in the field.

DEF. STRENGTH OF FIELD AT A POINT *is the magnitude of the force experienced by a unit of mass when placed at that point in the Field of Force.*

The Strength of Field is clearly the force per unit mass, and is the same numerically as the acceleration at the point. Thus the strength of the Earth's gravitational field at the level of the sea in latitude of Paris is 981, since a gramme placed there experiences 981 dynes, or units of force.

32. When we know at every point in a field of force the direction of the line of force and strength of the field our knowledge of that field of force is complete. We proceed to explain a system by which these can be expressed more concisely in terms of one quantity at each point—the Potential.

We have seen that if any two points A, B be taken in a field of force, and a unit of matter be carried from A to B

against the force in the field, a certain amount of work will be done on the unit, and if the unit of mass be allowed to pass back from B to A by frictionless constraint, the particle will acquire an equal amount of energy in its fall. The principle of conservation of energy shows that the amounts of work or energy will be the same, whatever path be pursued from A to B or from B to A respectively. For if more energy were acquired in falling along a path BCA, than along another path BDA, then by allowing the particle constantly to fall along BCA, and to return along BDA, we should have unlimited source of energy.

DEF. DIFFERENCE OF POTENTIAL AT ANY TWO POINTS *is the work done in carrying a unit of mass from one point to the other and depends only on the positions of the two points in the field.*

Thus the difference of potential for two places near the Earth's surface will be expressed in foot-pounds or cm.-gms. respectively by $32{\cdot}2\,h$ or $981\,h$, where h is the difference in height above the sea-level in feet or cms., the force of gravity being assumed that at the sea-level about the latitude of London. The standard from which h is measured is plainly arbitrary, as we are only concerned with the difference—practically it would be the sea-level, and we might speak strictly of places above the sea-level as having positive, and below the sea-level negative potential.

Though heat does not, as far as at present known, fall under the category of physical forces we are now considering, the quality called temperature is strictly analogous to potential. When we speak of the temperature of a certain place near a hot body, we only give the difference between its thermal condition and that of ice-cold water, the ice-cold water being a standard arbitrarily chosen on account of its convenience of reproduction. Or if the measure is given in Fahrenheit degrees, it is referred to the standard of a mixture of salt and snow. Generally we are no more able to give an absolute measure of potential than we are able to protrude our thermometer bulbs into interstellar space to find an absolute zero of temperature. But no confusion will arise if we keep before us that we are not giving the potential at a point *absolutely*,

but only the difference between the potential at that point and at another we have before agreed upon. The absolute zero is often spoken of as that at an infinite distance from all attracting matter, just as the absolute zero of temperature is that of interstellar space.

DEF. ZERO, POSITIVE AND NEGATIVE POTENTIAL. *Zero Potential is the potential at a certain point chosen as a standard of reference. Any place which requires work to be done to bring the unit of mass from the zero point to it will have positive potential. And any place which requires work to be done to bring the unit of mass from it to the zero point will have negative potential.*

33. We can express conveniently the component, in any direction, of the force on unit mass at a point in terms of the variation of potential along this direction.

Take any two points, A, B, in the field, and call their difference of potential V. Then if F be the average force between A and B resolved along AB, we have $F \times AB = V$, or $F = \frac{V}{AB}$. Hence the average force along any line will be given by the average rate at which potential changes along the line, and if the line be made very short, we may say that the force, in that direction, is the change of potential per cm. at that point in the given direction. Since the resultant force at the point is given by the direction of the line of force, and the force in any other direction will be the component in that direction of the resultant force, it follows that the potential changes most rapidly along the line of force, and less and less rapidly in directions more and more inclined to the line of force, while in a direction at right angles to the line of force the rate of change of potential must vanish.

This may be illustrated from gravitation. At any point on an inclined surface the line of force will be the line of greatest slope, or the line along which potential changes most rapidly, while perpendicular to this line will be a horizontal line or line of no change of potential.

34. If a surface be drawn through the field of force, which everywhere cuts at right angles lines of force, the rate of change of potential along it will be zero, or the surface will be an equipotential surface, and the force resolved along it at any point will always vanish, so that no work is done in moving matter along such a surface. Due to the Earth's gravitation the surface of the sea is an equipotential surface, as also the surface of any plain as determined by the spirit level or a plumb line.

DEF. EQUIPOTENTIAL SURFACE *is a surface drawn through all points in the field at which the potential is the same, and it everywhere cuts lines of force at right angles.*

It is clear that different equipotential surfaces cannot intersect, for at their intersection the potential would have two different values, and lines of force would also intersect.

35. The foregoing definitions and propositions contained in them are applicable to any Conservative System, that is to any system of forces to which Newton's Laws of Motion are applicable. We now proceed to some special applications of them in the particular case of forces such as occur in nature.

It is found by experiment and observation that between every two particles of matter in the universe there exists an attraction, which depends only on the masses of the particles and on the distance between them. As the distance increases, the force of attraction diminishes according to a law called the law of inverse squares. Thus if the distance be doubled, the force is reduced in the ratio 1 to 4, or $\frac{1}{2^2}$; if the distance be trebled, the force is only one-ninth or $\frac{1}{3^2}$ of its initial value, and so on. This is expressed by saying that if the masses are m and m', and the distance r, the force of attraction between them is $\frac{mm'}{r^2}$. It is proper to note here that the units of measure are different from to those on the C. G. S. system, for the latter would require that the attraction between

two unit masses placed a cm. apart should be a dyne, which is certainly not the case. The units in which these quantities are measured are those of the universal gravitational system. In this system the measure of the acceleration produced by mass m on a particle at distance r is $\frac{m}{r^2}$. We shall now investigate two very important cases of attraction coming under this law.

36. Prop. I. To find the strength of field due to a thin circular plate at a point in a line perpendicular to it through its centre.

Let us suppose the plate divided into very narrow circular rings, drawn about its centre A (Fig. 6), and let PQ be a type of such rings. We shall consider the attraction of each ring separately, and compound them to find that of the whole plate. Let O be the point at which a unit of mass is placed on which we are required to find the attraction of the plate. Join OA. The resultant attraction will be by symmetry along OA. We shall therefore resolve each force along that line, and add together the resolved parts so obtained.

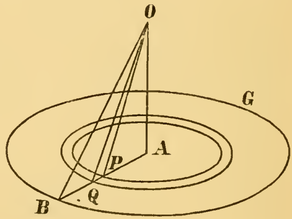

Fig. 6.

Now all parts of each ring are at the same angular distance from OA and at the same distance from O. Let the radius AB cut one ring as at P and Q, where P, Q are points on its inner and outer edge respectively. Join OB, OQ, OP.

If we take an element of the ring of mass m, its attraction lies between $\frac{m}{OP^2}$ and $\frac{m}{OQ^2}$, which it would be were the mass collected at P or Q respectively. *We shall assume the attraction to be* $\frac{m}{OP \cdot OQ}$. Again the direction of this resultant attraction will lie between OP and OQ. *We shall assume it to be towards a point R in PQ such that*

$$OR = \tfrac{1}{2}(OP + OQ).$$

Thus the attraction resolved along OA of the element, on the unit mass at O

$$=\frac{m}{OP\,.\,OQ}\,.\cos ROA=\frac{m}{OP\,.\,OQ}\cdot\frac{OA}{OR}.$$

Adding together all the elements of the ring, the attraction

$$=\frac{\text{mass of ring}}{OP\,.\,OQ}\cdot\frac{OA}{OR}.$$

But the mass of the ring (ρ being the mass of a unit of area)

$$\begin{aligned}&=\pi\rho\,(AQ^2-AP^2)\\&=\pi\rho\,(OQ^2-OP^2)\\&=\pi\rho\,(OQ+OP)(OQ-OP);\end{aligned}$$

$\therefore$ the attraction of the ring

$$\begin{aligned}&=\frac{\pi\rho\,(OQ+OP)(OQ-OP)}{OP\,.\,OQ}\cdot\frac{OA}{OR}\\&=2\pi\rho\,.\,\frac{OQ-OP}{OP\,.\,OQ}\,.\,OA,\text{ since } OR=\tfrac{1}{2}(OP+OQ)\\&=2\pi\rho\,.\,OA\,.\left(\frac{1}{OP}-\frac{1}{OQ}\right).\end{aligned}$$

Now suppose AB to be divided up into a very large number n of such rings, which cut AB in $P_1, P_2, P_3, \ldots P_{n-1}$. To each of such rings the above formula applies, and the attraction of the whole plate which equals the sum of the attraction of all the separate rings

$$\begin{aligned}&=2\pi\rho\,.\,OA\,.\left[\left(\frac{1}{OA}-\frac{1}{OP_1}\right)+\left(\frac{1}{OP_1}-\frac{1}{OP_2}\right)\right.\\&\qquad\left.+\left(\frac{1}{OP_2}-\frac{1}{OP_3}\right)+\ldots+\left(\frac{1}{OP_{n-1}}-\frac{1}{OB}\right)\right]\\&=2\pi\rho OA\left(\frac{1}{OA}-\frac{1}{OB}\right)\\&=2\pi\rho\left(1-\frac{OA}{OB}\right)\\&=2\pi\rho\,(1-\cos\alpha),\end{aligned}$$

where α = half the angular diameter of the plate as seen from O.

If the plate be of very large extent or the particle at O very near to it, α will become very nearly a right angle, and its cosine will be so small that it may be neglected compared with unity. Hence the attraction of any plate on a unit-mass at a distance from it, very small compared with its diameter, is always $2\pi\rho$.

In the above process two assumptions are made which are italicised.

It remains for us to show that the error in each element cannot on summation mount up to a significant term in the result.

Both assumptions consist in assigning to a term a value intermediate between the extreme values, of which the geometry showed it capable.

The error therefore in the case of any element cannot exceed the difference of these extreme values. Hence the whole error in estimating the attraction of the ring PQ is less than

$$\frac{\text{mass of ring}}{OP^2} \cdot \frac{OA}{OP} - \frac{\text{mass of ring}}{OQ^2} \cdot \frac{OA}{OQ},$$

or
$$< \pi\rho \,.\, OA \,.\, (OQ^2 - OP^2)\left(\frac{1}{OP^3} - \frac{1}{OQ^3}\right),$$

or $$< \pi\rho \,.\, OA \,\frac{(OQ^2 - OP^2)(OQ - OP)(OQ^2 + OP \,.\, OQ + OP^2)}{OP^3 \,.\, OQ^3};$$

or much more

$$< \pi\rho \,.\, OA \,.\, \frac{(OQ^2 - OP^2)(OQ - OP)(OQ + OP)^2}{OP^3 \,.\, OQ^3},$$

or
$$< \pi\rho \,.\, OA \,.\left(\frac{1}{OQ} + \frac{1}{OP}\right)^3 (OQ - OP)^2.$$

But OP and OQ are both greater than OA.

Hence error $< \pi\rho OA \left(\dfrac{2}{OA}\right)^3 (OQ - OP)^2$.

Let now the n rings be chosen so that $OQ - OP$ is the same for each, so that $n(OQ - OP) = OB - OA$.

Hence whole error $< n\pi\rho OA \left(\frac{2}{OA}\right)^3 (OQ - OP)^2$,

or $$< \frac{8\pi\rho}{OA^2} . (OB - OA) . (OQ - OP).$$

Now the number of the rings can be made as great as we please, and therefore $OQ - OP$ can be made in all cases indefinitely small, and it is clear that the whole error committed cannot exceed a quantity itself indefinitely small, which may therefore be neglected. The proposition is now completely established.

37. NOTE. As the method used in the above Article will enter largely into our future investigations it may be expedient to give here a general statement of the nature of the process.

We have generally to sum a series of the form

$$u_0 + u_1 + u_2 + \ldots + u_{n-1},$$

where all we know about the successive terms is that they are certain very small quantities each of which lies between very narrow limits, defined geometrically: while all we know about n is that it is a very large number. Our method consists in putting u_0, u_1, u_2, &c. in the form

$$u_0 = x_0 - x_1,$$
$$u_1 = x_1 - x_2,$$
$$u_2 = x_2 - x_3,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$u_{n-1} = x_{n-1} - x_n.$$

Hence on addition the sum of the series

$$= x_0 - x_n.$$

Now in transforming u_0 for instance into $x_0 - x_1$, we generally take for u_0 any value between its extreme values which can easily be decomposed into the form indicated. It is necessary to show that no appreciable error is introduced into the result. Suppose $\frac{a}{b}k$, $\frac{a'}{b'}k$ to be the extreme values of which one of the terms u is susceptible. Here $\frac{a}{b}$ and $\frac{a'}{b'}$ are

fractions whose numerator and denominator differ by small quantities easily expressed as multiples of k, so that we can assume $a' = a + pk$, $b' = b + qk$; and k is itself a very small quantity. Hence the error committed in this term cannot exceed *numerically*

$$\left(\frac{a}{b} - \frac{a+pk}{b+qk}\right) k,$$

which we will for simplicity assume positive. The error therefore is certainly less than

$$\frac{aqk - bpk}{b(b+qk)} \cdot k,$$

or $$< \frac{aq - bp}{bb'} \cdot k^2.$$

Let us now choose the successive terms so that k may be the same for each and $nk = K$, some finite quantity. Also on this hypothesis let $\dfrac{aq-bp}{bb'}$ have its greatest possible value, C suppose, which will certainly be finite as neither b nor b' vanishes. Then the whole error will certainly be less than nCk^2 or CKk.

But by making the number of terms sufficiently great k can be made indefinitely small, and hence the term CKk is also indefinitely small.

This shows that the error committed can in no case rise to importance in the final summation.

The same reasoning holds good if the value assumed for u does not lie between its extreme values, provided the greatest possible error be some finite multiple of k^2.

The following applications of this method will be commonly applied hereafter:

(i) If k represent a small angle we shall assume that its sine, circular measure, and tangent are interchangeable.

(ii) If k represent a small arc we shall assume that the chord may be substituted for the arc and *vice versa*.

(iii) If k be a small fraction we shall assume that we may

substitute for it any convergent algebraical series whose first term is k, as for instance $\log(1+k)$ or $-\log(1-k)$.

The student of Calculus will at once see that these sums are in reality definite Integrals, and the terms rejected are terms of the second order in the differential.

38. Prop. II. A uniform spherical shell exercises no attraction on a particle placed in its interior.

Suppose a unit mass to be placed at O, a point in the interior of the shell.

Then draw through O a double cone of small vertical angle. The intersection of the cone with the shell cuts off two small frusta AB', CD' from the cone. The attractions of these two small elements of the shell, on the particle at O, are exerted in opposite directions, and the resultant attraction towards A is

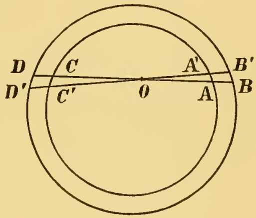

Fig. 7.

$$\frac{\text{mass of } AB'}{OA^2} - \frac{\text{mass of } CD'}{OC^2}.$$

But since the tangents drawn to the sphere AA', CC' are equally inclined to AOC, we may consider AB' and CD' as parallel plates of equal thickness cut from a cone, and in this case

$$\frac{\text{volume of } AB'}{\text{volume of } CD'} = \frac{OA^2}{OC^2},$$

and the volumes are proportional to the masses, since the shell is homogeneous;

$$\therefore \frac{\text{mass of } AB'}{\text{mass of } CD'} = \frac{OA^2}{OC^2},$$

$$\therefore \frac{\text{mass of } AB'}{OA^2} = \frac{\text{mass of } CD'}{OC^2}.$$

Hence the resultant attraction of the two opposite elements on O is nil.

Now if the whole shell be cut up into similar pairs of

elements, the same reasoning will hold good for each pair, and the whole attraction of the shell on any internal point vanishes.

Since there is no force within the spherical shell the rate of change of potential is nil, or the potential is the same throughout the interior of the shell.

39. **Prop. III. To find the work done in carrying a unit of mass against the attraction of any system of particles from one point to any other point, or to find the difference of potential between two given points.**

Let PQ be a small element of the path pursued, and let a mass of matter, m, be placed at O. Join OP, OQ, and from Q draw QR perpendicular to OP.

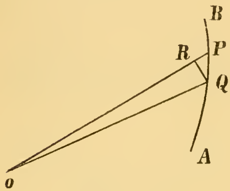

Fig. 8.

Then the attraction on the unit mass anywhere between P and Q is represented by $\frac{m}{OP \,.\, OQ}$ for the reason given in Prop. I.

This attraction resolved along PQ

$$= \frac{m}{OP \,.\, OQ} \cos OPQ$$

$$= \frac{m}{OP \,.\, OQ} \cdot \frac{PR}{PQ}.$$

Hence the work done in carrying the unit of mass from Q to P is

$$\frac{m}{OP \,.\, OQ} \cdot \frac{PR}{PQ} \cdot PQ$$

$$= \frac{mPR}{OP \,.\, OQ} = \frac{m(OP - OQ)}{OP \,.\, OQ}$$

$$= m\left(\frac{1}{OQ} - \frac{1}{OP}\right).$$

If there be other particles in the system, it is clear that the force in direction PQ is equal to the sum of the separate forces. Hence work done from Q to P

$=$ (total force along PQ) $\times PQ$;

$=$ sum of work done against each separately.

Hence the work done against the attraction of a system of particles m_1, m_2, ... placed at points O_1, O_2, ... may be expressed by

$$\Sigma m\left(\frac{1}{OQ}-\frac{1}{OP}\right).$$

In the same way, dividing the whole arc into similar elements, and performing the summation, we find that the work done against any attracting system in carrying unit mass from A to B

$$=\Sigma m\left(\frac{1}{OA}-\frac{1}{OB}\right)$$

$$=\Sigma m\left(\frac{1}{r}-\frac{1}{R}\right).$$

This shows us that the whole work done against any attracting system in moving a body from A to B, along any path whatever, is the sum of work done along the whole path against each element of the system taken separately; and we see it to be independent of the path pursued from A to B.

Cor. 1. If the particle move freely from B to A under the influence of the attracting system, the law of Kinetic energy must hold (see Art. 28), and we have, if v, V be the final and initial velocities,

$$\frac{1}{2}Mv^2-\frac{1}{2}MV^2=M\Sigma m\left(\frac{1}{r}-\frac{1}{R}\right),$$

where M is the mass of the particle moved, and m is, as above, the mass of one of the attracting particles.

Cor. 2. Let us suppose one of the points (B suppose) to be at an infinite distance or at the absolute zero of potential (see Art. 32). We then get for the absolute potential at any point A of a system of attracting particles, the expression $\Sigma\frac{m}{r}$. Strictly we must change the sign of this expression to give the potential at a point, since work has to be done in carrying the unit mass from the point to another point at which the potential is made zero. There will be no confusion in omitting the negative sign if we make the convention

that work is done against gravitational force when the unit mass is carried from a place at numerically higher to one at numerically lower potential.

COR. 3. It is easily seen that the rate of change of potential in any direction gives us the resolved strength of the field in that direction. For taking a single particle m at O (Fig. 8), the potentials at Q, P of m are $\frac{m}{OQ}, \frac{m}{OP}$. Hence the change of potential along PQ

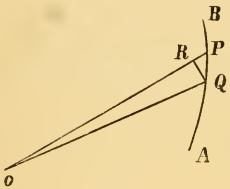

Fig. 8.

$$=\frac{m}{OQ}-\frac{m}{OP}=\frac{m(OP-OQ)}{OP.OQ},$$

and therefore the rate of change along this line

$$=\frac{m}{OP.OQ}\cdot\frac{OP-OQ}{PQ},$$

which when P, Q are very near together becomes $\frac{m}{OP^2}\cdot\frac{PR}{PQ}$ or $\frac{m}{OP^2}\cos QPR$, the resolved part of the force due to O along PQ, and the same will be true for each element of which the potential is made up.

40. As a concrete illustration of these principles, suppose the field of force due to a single particle of attracting matter. The lines of force are straight lines emanating from this point; the strength of a field at a distance r from the attracting point m is $\frac{m}{r^2}$ in a line towards the attracting point; the potential at the same point is $\frac{m}{r}$, and the equipotential surfaces, over each of which $\frac{m}{r}$ must be constant, will be surfaces for which r is constant or a system of spheres having the attracting point for centre.

We have already noticed that since there is no force exerted by a spherical shell on a particle inside it, the

potential everywhere within it must be the same, and will therefore equal the potential at the centre of the sphere. Since every particle of the shell is at the same distance (radius) from the centre, the potential at the centre

$$=\Sigma\frac{m}{r}=\frac{\Sigma m}{r}=\frac{\text{mass}}{\text{radius}},$$

which is therefore the value of the potential within every uniform thin spherical shell.

41. If we have a mass m_1 at a point A_1 distant r_1 from a certain point B_1, then the potential due to A_1 at B_1 is $\frac{m_1}{r_1}$. This quantity is a measure of the work done in taking a unit mass from B_1 to an infinite distance from A_1 or out of A_1's field of force. Let there be at B_1 a mass μ_1, then the work done in moving this mass out of A_1's field of force will be $\frac{m_1\mu_1}{r_1}$, and this expression may be called the potential of m_1 on μ_1. The symmetry of its form shows that it is also the potential of μ_1 on m_1. If there be a system of points $A_1, A_2\ldots$ at which are masses $m_1, m_2\ldots\ldots$ distant $r_1, r_2\ldots\ldots$ from μ_1, then $\mu_1\Sigma\frac{m}{r}$ will represent the potential of the system of particles A on μ_1. Let there now be another system of particles $\mu_2, \mu_3\ldots\ldots$ at points $B_2, B_3\ldots\ldots$ and let the value of $\mu\Sigma\frac{m}{r}$ be computed for each element of B, then the sum denoted by $\Sigma\mu\Sigma\frac{m}{r}$ will give the whole potential of the system A on the system B, or the whole work done in carrying the system B out of the field of force due to A, and this may be defined to be the potential of A on B. The process by which this value is obtained is clearly the same as joining every m in A with every μ in B, measuring their distance r, and adding up all such terms as $\frac{m\mu}{r}$. This sum may be written $\Sigma\Sigma\frac{m\mu}{r}$, remembering that both systems must

be exhausted in making the summation. The form of this latter expression, or indeed the Third Law of Motion, shows that it must be identical with that obtained by computing the potential of B on A. We conclude that the potential of A on B is the same as that of B on A, and its value may be termed the Mutual Potential of A and B.

DEF. POTENTIAL OF A SYSTEM A ON A SYSTEM B: MUTUAL POTENTIAL. *The potential of a system A on another system B of attracting matter is the work done in carrying B out of the field of force due to A, and is equal to the potential of B on A. This function, spoken of in reference to the systems A and B, may be called their Mutual Potential.*

42. Since lines of force exist throughout a field of force but do not intersect, if we draw in the field any closed curve and draw lines of force through every point in it, we shall have a tubular surface bounded by lines of force which is called a Tube of Force.

DEF. TUBE OF FORCE *is any tubular surface such that the line of force through every point on it lies wholly in that surface.*

In the case of a single attracting particle referred to above the tubes of force will be cones having the attracting particle at their vertex.

43. We now proceed to investigate some very important properties of Tubes of Force, taking first the case of a small tube due to a single attracting particle.

Prop. IV. If a cone of very small vertical angle be drawn having a particle of attracting matter at its vertex: if F be the attraction on unit mass at any point within the cone computed in any direction, and S the area of the section of the cone perpendicular to the direction of F, then the product FS is constant throughout the cone.

(i) Let the direction of the force be along the axis of the cone; the sections are then at right angles to the cone.

Let P, Q be two points on the axis of the cone, and F_1, F_2 the attractions exerted by O on them; S_1, S_2 the

sections of the cone through P, Q perpendicular to the axis.

Then

$$F_1 : F_2 :: \frac{m}{OP^2} : \frac{m}{OQ^2},$$

where m is the mass of the attracting particle at O.

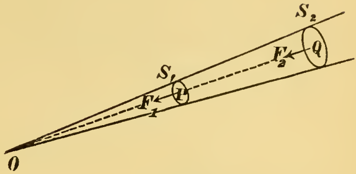

Fig. 9.

And since S_1, S_2 are similar figures,

$$S_1 : S_2 :: OP^2 : OQ^2,$$

$$\therefore F_1 S_1 : F_2 S_2 :: m : m;$$

hence in this case $F_1 S_1 = F_2 S_2$.

(ii) Let the force be inclined to the axis of the cone at any angle θ, the section is then oblique and inclined at an angle θ to the right section.

Let F and Aa (S) be the resultant force and right section at any point P: also let F_1, S_1 be the same quantities for an oblique section Bb through the same point.

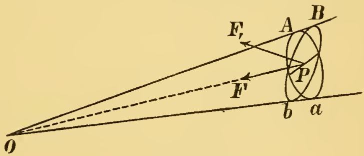

Fig. 10.

Then we may regard Aa as the orthogonal projection of Bb, and the inclination of the two sections being θ, we have

$$Bb \cos\theta = Aa,$$

$$\text{or, } S_1 \cos\theta = S.$$

Again, since F_1 is the resolved part of F in a direction inclined at an angle θ,

$$F_1 = F\cos\theta;$$

$$\therefore F_1 S_1 \cos\theta = FS\cos\theta,$$

$$\text{or, } F_1 S_1 = FS,$$

which proves the proposition generally.

44. Prop. V. If the area of a closed surface be divided into a large number of elements σ_1, σ_2, σ_3..., and the force on unit mass, due to an attracting system outside it, be computed over each elementary area, normal to it reckoned outwards, the sum denoted by the symbol $\Sigma F\sigma$ shall vanish.

Take one element of the attracting matter, as O, and draw from O a small cone which cuts the surface in two elementary areas σ, σ', and let F, F' be the forces normal to the surface computed outwards. Then by the preceding proposition

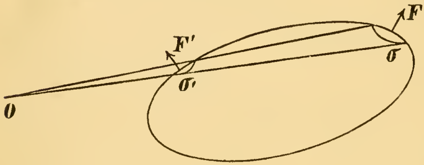

Fig. 11.

$$-F\sigma = +F'\sigma':$$

the sign $-$ being attached because the normal components at σ and σ' are in opposite directions with respect to O, that is, one tends *to* O and the other *from* O;

$$\therefore F\sigma + F'\sigma' = 0;$$

and since the whole surface can be cut up into similar pairs of elements we have over the whole surface $\Sigma F\sigma = 0$.

Again, what is true for each particle of the attracting mass outside, taken separately, is true when they are all taken together. Hence, if F_1, F_2, &c. represent the resultant force due to the whole external mass on each element of the surface, we must also have over the whole surface as before $\Sigma F\sigma = 0$.

This proposition, as well as substantially the proof here given, is due to Sir G. G. Stokes.

45. Prop. VI. If a tube of force, bounded as to its ends by two equipotential surfaces, have the ends divided into elements σ_1, σ_2, σ_3, &c., σ_1', σ_2', σ_3', &c., and the resultant force on unit mass, computed over each element, F_1, F_2, F_3, &c., F_1', F_2', F_3', &c.; then

$$\Sigma F\sigma = \Sigma F'\sigma'.$$

For the tube of force so bounded is a closed surface, and we may apply to it the statement of the preceding proposition.

Now since the tube of force is bounded by lines of force, and since a force can produce no effect in a direction at right angles to itself, the component of the force perpendicular to the surface at every point on the tubular surface is zero.

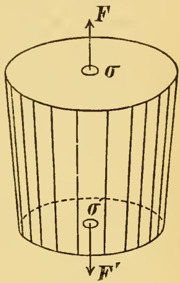

Fig. 12.

Hence we have only to consider the force on the ends of the tube, that is on the equipotential surfaces, and we have

$$\Sigma F\sigma - \Sigma F'\sigma' = 0,$$

the $-$ sign being used because the direction of the force on one surface is inwards and on the other outwards, with respect to the portion of the tube of force under consideration;

$$\therefore \Sigma F\sigma = \Sigma F'\sigma'.$$

If the force is uniform over the ends or equipotential surfaces, then we may write the equation as

$$FS = F'S',$$

where S is the whole area of one end or cross section and S' of the other, and F, F' are the forces over each respectively.

The same law can be extended, just as in Art. 43, to the case of a surface cutting the tube obliquely. For if F, S represent the force and area of cross section when perpendicular to the tube, F_1, S_1 similar things when oblique to the tube, we have

$$S = S_1 \cos\theta,$$

$$F_1 = F\cos\theta;$$

$$\therefore F_1 S_1 = FS.$$

46. **Prop. VII. If a small tube of force cut through a thin plate of attracting matter perpendicular to it, the product $F\sigma$ in passing from one side to the other changes by $4\pi m$, where m is the mass of matter included in the tube.**

For let AB be the thin plate of matter included, and let P and Q be two points taken very near the plate and on opposite sides. We will denote all the attracting matter outside the plate by M, the tube of force being due jointly to the attraction of M and the plate AB. Since the forces due to M and to AB are at P and Q both perpendicular to AB,

we may by the Second Law of Motion consider their effects separately and add them together.

Let F' be the resultant attraction of M on unit mass at P or Q which are indefinitely near together, the direction of F' being assumed in the figure from Q to P. The attraction of the plate will be $2\pi\rho$ (Art. 36). At P the attraction of the plate acts *against* F', and hence the complete product $F\sigma$

$$= (F' - 2\pi\rho)\,\sigma.$$

At Q the attraction of the plate acts *with* F', and hence the complete product $F\sigma = (F' + 2\pi\rho)\,\sigma$.

Hence the change in the product $F\sigma$ on passing from one side of the plate to the other

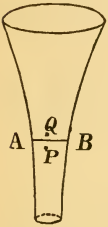

Fig. 13.

$$= (F' + 2\pi\rho)\,\sigma - (F' - 2\pi\rho)\,\sigma$$
$$= 4\pi\rho\sigma$$
$$= 4\pi m,$$

since where ρ = mass per unit area and σ the area of the plate, $m = \rho\sigma$.

The same proposition holds true if the mass be not a thin plate, since we may conceive it to be made up of thin plates cut perpendicularly by tubes of force with spaces between; the above proposition is true for each plate separately, and consequently it is true for any mass cut through by a small tube of force.

NOTE. It must be carefully observed that the quantity denoted by F in these propositions is not of the nature of force, but is the *force per* unit of mass or the strength of the field, since there is clearly no such thing as resultant force at a point in a field of force, unless matter be brought there.

47. Prop. VIII. If one equipotential surface be known and the law of distribution of force over it, all other equipotential surfaces can be drawn by pure geometry only.

Let the surface be mapped out into small areas such that the product of each area into the force near it shall be constant. These areas will form the bases of tubes of force, and for each tube of force the product $F\sigma$ remains the same.

Since the work done in carrying a gramme from one

surface to the next will be constant (V), the distance x along a normal from one surface to the next will be given by

$$Fx = V.$$

Hence, to pass from one surface to the next, we remember that

$$Fx = V \text{ and } F\sigma = C;$$

it follows that x is in a constant ratio to σ, and we have round each tube of force to raise normals proportional in length to the area of that tube, and the locus of the extremities of these normals will be the next equipotential surface. Similarly, by using the projections of the tubes of force on the new equipotential surface, we can pass on to the surface next to it, and so on indefinitely.

It will be observed that, in accordance with the last proposition, when any tube of force passes through attracting matter the product $F\sigma$ is changed by $\pm 4\pi m$, which will correspond to increasing or diminishing the number of tubes on the equipotential surface.

48. Prop. IX. The potential value in free space is never discontinuous, that is it never at a point or surface makes a sudden change from one value to another.

For the difference of potential at two points AB is the work done on a gramme carried from A to B, hence if V be the potential difference, and F the average force along the line, $F \times AB = V$. Hence if V has a finite value when AB is zero (which corresponds to a discontinuity), F must be infinite, which never happens in free space.

Cor. In the same way the potential cannot be constant on one side of a given surface and vary on the other side unless the surface has attracting matter distributed over it. For the surface bounding the region of constant potential must be equipotential, and tubes of force must cut it at right angles. But on the side of the surface on which potential is constant $F = 0$, and $F\sigma = 0$. Hence $F\sigma = 0$ throughout the tube, and therefore $F = 0$ or the potential is constant on the opposite side of the surface also.

49. Prop. X. The potential never has a maximum or minimum value at a point in free space.

For if it had, the potential at a certain point would be rather greater or less than at all points round it, and the equipotential surfaces would degenerate to a single point. Tubes of force starting from that surface or point would have zero bases and, unless F were infinite at the point, the product $F\sigma$ would equal 0 through all space; and F cannot be infinite at a point in free space.

50. Prop. XI. No particle can be in stable equilibrium under the attraction of a material system.

If possible let O be the point of equilibrium and P, Q points on opposite sides of it. Draw any line POQ. Then since O is not a point of maximum or minimum potential, (Art. 49) the potentials at P, O, Q are generally in ascending or descending order of magnitude. Thus the force along POQ is on one side of O towards it and on the other away from it, a condition inconsistent with stable equilibrium. For particular directions of the line POQ, corresponding to constrained motion of the particle, neutral or unstable equilibrium would be possible. Thus if POQ were along an equipotential surface, the rate of change of potential at O vanishes, or the equilibrium is neutral, and again if the force vanish at O there would be equilibrium though unstable.

51. Prop. XII. At great distances from an attracting system the equipotential surfaces tend to become spheres about the centre of gravity of the system, the force over each sphere being uniform.

Suppose G the centre of gravity of the attracting system, and A_1 an element of it distant a_1 from G, and let P be the distant point at which we wish to calculate the potential. By a distant point we mean a point such that squares and higher powers of the fraction $\dfrac{GA_1}{GP}$ may be neglected compared to unity.

Let $$GA_1 = a_1, \quad GP = R, \quad A_1P = r_1,$$
and $\angle PGA_1 = \theta$, and let the mass of element at A_1 be m_1.

Then potential at P due to $m_1 = \dfrac{m_1}{r_1}$.

But

$$r_1^2 = R^2 + a_1^2 - 2Ra_1 \cos\theta;$$

$$\therefore r_1 = R\left(1 - \frac{a_1}{R}\cos\theta\right)$$

neglecting $\left(\dfrac{a_1}{R}\right)^2$;

$$\therefore \frac{1}{r_1} = \frac{1}{R}\left(1 + \frac{a_1}{R}\cos\theta\right)$$

to the same degree;

$$\therefore \frac{m_1}{r_1} = \frac{m_1}{R} + \frac{m_1}{R^2}a_1\cos\theta.$$

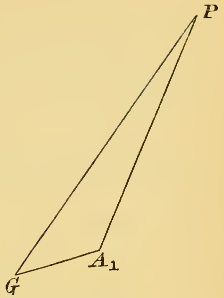

Fig. 14.

Hence, for the whole system,

$$\Sigma\frac{m}{r} = \frac{\Sigma m}{R} + \frac{1}{R^2}\Sigma ma\cos\theta.$$

But $a_1 \cos\theta =$ projection of GA_1 on GP, and therefore, by the principle of centre of gravity, $\Sigma ma \cos\theta = 0$;

$$\therefore \Sigma\frac{m}{r} = \frac{\Sigma m_1}{R},$$

or the potential of the system is the same as if the whole system were collected at its centre of gravity. In this case the equipotential surfaces are spheres about the centre of gravity as centre, and the force over each sphere is uniform.

Examples on Chapter II.

1. Show that in computing the attraction of a solid sphere on a point within its mass we may neglect all of the sphere more remote from the centre than the given point.

2. Given that the volumes of spheres are proportional to the cubes of their radii, show that the attractions exerted by a sphere on points within it are directly proportional to their distances from the centre.

3. Show that, supposing the density of the earth to be uniform and its diameter doubled, the acceleration at its surface would be double its present value.

4. If three particles of masses m_1, m_2, m_3 be placed at the angular points of a triangle, the potential at the centre of the circumscribing circle is $\frac{m_1+m_2+m_3}{R}$, where R is the radius of the circle.

5. If any number of particles be distributed over the surface of a sphere, the potential at the centre of the sphere is $\frac{\Sigma m}{R}$, where Σm is the sum of all the masses and R the radius of the sphere.

6. At the angular points of a triangle are placed masses equal numerically to the lengths of the opposite sides. Show that the potential at the intersection of perpendiculars is equal to $\tan A \tan B \tan C$; at the centre of the circumscribed circle it is $8 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$, and at the centre of the inscribed circle

$$\frac{8R}{r}\left\{\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4} - \sin\frac{\pi-3A}{4}\sin\frac{\pi-3B}{4}\sin\frac{\pi-3C}{4}\right\},$$

where R and r are the radii of circumscribed and inscribed circles.

7. Calculate the potential of a circular plate on a point situated on a line through its centre perpendicular to its plane.

Ans. $2\pi\rho l(1-\cos\alpha)$; if l be distance from edge and 2α the angle subtended by the plate.

8. Show that in a field of uniform force the lines of force are parallel straight lines and the equipotential surfaces a system of parallel planes.

9. Show that if the equipotential surfaces be a system of concentric spheres the force over each sphere is uniform.

10. Show that the attractions of all parallel plates of equal thickness cut from a right cone on a particle placed at the vertex are equal.

11. Show that the last proposition is true for sections taken from an oblique cone.

12. Two similar right cones of like material attract equal particles placed at their respective vertices. Prove that the attractions are proportional to the heights.

13. Find the potential of a solid right cone on a particle at its vertex.

Ans. $\pi\rho l^2 \cos\alpha\,(1-\cos\alpha)$; if l be the slant height and α the semi-vertical angle.

14. If particles be placed at the middle points of the sides of a triangle, their masses being numerically the same as the sides, show that the potential at the centre of the circumscribing circle is $2\tan A \,.\, \tan B \,.\, \tan C$.

15. If equal particles of matter be placed round an ellipse at distances such that the angle subtended between any two successive particles at the focus is constant, show that the potential at the focus is $\dfrac{nm}{l}$, where n is the number of particles, m the mass of each, and l the semi-latus rectum.

Note. Using the polar equation $\left(\dfrac{l}{r}=1-e\cos\theta\right)$ the proposition follows at once.

16. Find the potential of a very narrow circular annulus at its centre.

17. Find the potential of a broad circular annulus at its centre.

18. Find the potential of a sector of a circle at the centre of the circle.

19. If the equipotential surfaces be a system of confocal spheroids, show that the lines of force are systems of hyperbolas having the same foci.

CHAPTER III.

APPLICATION OF POTENTIAL TO STATICAL ELECTRICITY.

52. BEFORE proceeding to apply the properties of Potential to the investigation of Electricity, we must state briefly one or two of the experimental laws on which such application depends.

EXPERIMENT I. *There is no electrical force within a closed electrified conductor, unless there are other electrified bodies within it and insulated from it.*

This has been shown conclusively in numerous experiments devised by Faraday. Having tested by the proof plane, and Coulomb's balance, the inner surfaces of different conductors, of every variety of shape—spheres, cylinders, &c., with the outer surfaces either completely closed as with tinfoil, or closed only by a conducting network of wire gauze or of linen fibres, as in a butterfly net: he finally constructed a small house or room 10 or 12 feet cube, covered outside with tinfoil, and insulated on glass legs, so that the whole surface could be highly electrified by a powerful machine. Into this he carried various electroscopes, and with them applied the most delicate tests he knew of for electrification, without success, though the outside was highly electrified. Such was the delicacy of these tests that if there had been a ten-thousandth part of the electrification inside that there was outside he could not have failed to detect it.

This experiment is equivalent to saying that no field of electrical force exists within the substance of a conductor, and that therefore every electrical field is bounded by conducting surfaces, and consists wholly of non-conductors or dielectrics.

This experiment is true for surfaces under electrical induction as well as for those which are freely electrified.

53. EXPERIMENT II. *When a separation of electricities takes place by friction or any other means, the amounts of vitreous and resinous electricities produced are always such that, on being re-united, they exactly neutralize each other.*

This is shown clearly in any form of electrical machine in which the opposite poles are connected. For unless it were true one pole would, on working the machine, still acquire a charge of electricity.

DEF. COMPLEMENTARY DISTRIBUTIONS. *The two amounts which are produced when the electricities of a neutral body are separated, are said to be equal and of opposite sign, and we shall speak of them as complementary distributions.*

It is necessary in dealing with electricity to keep these complementary distributions in view, as we cannot in any problem have to deal with charges of vitreous or resinous electricity alone, but each charge has somewhere its complement of the opposite kind.

54. EXPERIMENT III. *Every electrified body when placed in a closed vessel whose surface is conducting, calls up by induction on the inner surface of the vessel a charge of electricity equal in quantity, but opposite in sign to its own charge.*

This is experimentally proved by Faraday's Ice-pail experiment. An electrified sphere is introduced into a hollow closed conductor, and the electricity induced on the *inside* by the charged body *before contact* of the sphere with the interior, is found *on contact* just to neutralize the body's charge. The complementary distribution on the outside, which is equal to the disguised or induced charge, must therefore be equal to the original charge of the body.

This experiment shows that when electrical experiments are performed in a room the complementary electrification always exists distributed over the walls of the room.

55. EXPERIMENT IV. *If two bodies be electrified and placed at a constant distance, great compared with their dimensions from each other; they exert on each other a force proportional to the product of the amounts of electricity they contain. This force is attractive if their electrification be opposite, repulsive if similar.*

We can measure the repulsion of two charged bodies by Coulomb's torsion balance, in which the moment of the repulsive or attractive force is equal to the torsion of the wire required to keep the bodies at a fixed distance.

The charges can be varied in the following manner: Provide a ball of the same size as the carrier and indicator balls of the torsion balance, insulated by a silk thread or gum-lac stem, which we shall call the discharging ball.

Having charged the carrier ball, it is placed in the balance; its charge is immediately divided equally with the indicator ball, and we can observe the torsion of the wire which keeps the two balls at any proposed distance apart: we have in this way a measure of the repulsive force between two quantities *each* $\frac{1}{2}$. We now remove the carrier ball, and divide its charge with the discharging ball, by which means the charge of each is reduced to $\frac{1}{4}$. The carrier ball is replaced in the balance, and the repulsion at the same distance observed.

The discharging ball is now discharged, the carrier ball removed and touched against the discharging ball, again replaced, and the repulsion at the same distance again observed.

By continuing this process we can observe the repulsion at fixed distances between quantities whose ratios are respectively 1 to 1; 1 to $\frac{1}{2}$; 1 to $\frac{1}{4}$; 1 to $\frac{1}{8}$; and so on. By this means it is found that, making certain allowances for loss of charge, the repulsion at a constant distance closely approximates to the law above given, and on increasing the distance, the law is found more and more nearly true.

By fixing a vertical wire in the balance to prevent the indicator ball from flying to the carrier ball, and first charging the carrier ball with negative electricity, the same law can be established for the attraction of oppositely electrified bodies.

56. EXPERIMENT V. *If constant charges of electricity be condensed in two points, and the distance between them varied, the force of attraction or repulsion is found to vary inversely as the square of the distance.*

This is shown by Coulomb's torsion balance also, by varying the distance of the conductors instead of the charges. It might also be inferred from the following considerations: Let a hollow spherical shell be charged with electricity. From its symmetry of shape it is clear that the distribution of electricity over it will be uniform, and the amount on any element of its surface therefore proportional to its area.

Now, referring to Art. 38, we see that if we wish to find the electrical force at a point O within an electrified sphere we divide the surface up into opposite pairs of elements AB', CD', and then assuming the law of inverse squares, prove that there is no force at that point. But if we suppose the law of the force unknown and call it the inverse nth power of the distance, the attraction exerted by the pair of opposite elements on O (see Fig. 7) will be (towards A)

$$\frac{\text{mass } AB'}{OA^n} - \frac{\text{mass } CD'}{OC^n}.$$

But $$\frac{\text{mass } AB'}{OA^2} = \frac{\text{mass } CD'}{OC^2} = k \text{ suppose.}$$

Hence the attraction on O *towards* A

$$= k\left(\frac{1}{OA^{n-2}} - \frac{1}{OC^{n-2}}\right).$$

If $n > 2$ and $OA < OC$ this result will be positive, or there will be an attraction towards the nearer side of the sphere. If $n < 2$ or negative the result is negative, showing that there will be a resultant attraction towards the more distant side of the sphere.

The whole attraction on the internal point can therefore only vanish when n exactly equals 2, i.e. for the law of the inverse square. The methods of detecting electrification are so much more delicate than any measurement by the torsion balance, that this constitutes the most reliable proof of the law, since we know as a fact that there is no electrical force anywhere within a closed electrified sphere. This is known as Cavendish's proof of the law of inverse squares of distances.

57. The results of the two preceding articles show that if we have quantities of electricity represented by q and q' placed at a distance r apart, the force between them varies as the product qq' and inversely as the square of r. Thus the force varies as $\frac{qq'}{r^2}$.

We now choose such an unit of electricity as to make the expression $\frac{qq'}{r^2}$ represent the actual force exerted. To do this we notice that when $q=1$, $q'=1$ and $r=1$, the expression reduces to unity. Then we have

DEF. UNIT QUANTITY OF ELECTRICITY. *The unit of electricity is such a quantity, that if condensed in a point it shall exert a unit of force, or one dyne, on another similar unit placed at a distance of one centimetre from it.*

58. The foregoing definition of unit quantity is abstract only, since no finite quantity of electricity can be condensed in a point. Electrification is always a property of a surface, and to express the degree in which a surface is electrified the term density is employed.

DEF. ELECTRICAL DENSITY. *Electrical density is a term used to denote the quantity of electricity on a surface per square centimetre.*

Thus if a surface of area s be electrified uniformly with a charge q, and if ρ be the density of the electrification,

$$\rho = \frac{q}{s} \quad \text{or,} \quad q = \rho s.$$

If the surface be not uniformly electrified we define density as the quantity which would be on a unit of area supposing the density uniform and of the same value as at the point under consideration.

59. It is usual to refer to the electricity which appears on the plate and prime conductor of an electrical machine as *positive*, while that which appears at the same time on the rubber or negative conductor is called negative. The propriety of these terms appears if we remember (Exp. II.) that

the amounts developed always neutralize each other. This is conveniently expressed algebraically by saying that if quantities q and q' of electricity are developed from a neutral body by friction or otherwise $q+q'=0$ always. Further, we have shown that if two similarly electrified particles containing quantities q and q' of electricity (both + or both −) be placed at a distance r from each other, there is between them a repulsive force measured by $\frac{qq'}{r^2}$, while if the quantities q and q' be one positive and the other negative, there is an attractive force measured by $\frac{qq'}{r^2}$. Generally we may say that between any two quantities q and q' there is a repulsive force, remembering that when the product qq' is negative the repulsion becomes negative, and negative repulsion is the same as attraction.

60. Having established the fundamental proposition that between two quantities q and q' of electricity condensed in points at a distance r from each other there is a force measured by $\frac{qq'}{r^2}$ which is repulsive if this product be *positive,* and attractive if it be *negative;* we can apply all our propositions on Potential of attracting matter to Potential of an electrical distribution.

We shall only have to substitute in our original definitions the *unit quantity of positive electricity condensed in a point* for the *unit mass.* This will be referred to as a plus-unit (written + unit).

Our Definitions for the electrical field will then be as follows :—

1. *Field of Electric Force* is the dielectric by which any positive electrification is separated from the complementary charge. This is clear since if the dielectric did not exist the Field would instantaneously be reduced to zero by conduction.

2. *Lines of Force* are lines in the field such that the tangent at each point shows the direction in which an electrified particle placed there would be urged by the Electric

Force. Since a positively and negatively electrified particle will be urged in opposite directions along the line of force, it is convenient to define the positive direction of the Line of Force as that direction in which a positively electrified particle will be urged.

3. *Strength of Field at a point*, or as it is often expressed, *Electric Force at a point*, is the force on a particle charged with a unit of positive electricity if placed at the point.

4. *Difference of potential at two points* in the field is measured by the work done in carrying a + unit from one point to the other in the field and is independent of the path pursued from one point to the other.

If the electrical system consist of electrified particles at points A_1, A_2 ...... holding quantities of electricity q_1, q_2...... respectively, and if there be two points P, Q so that $PA_1 = r_1$, $PA_2 = r_2$ ......, $QA_1 = R_1$, $QA_2 = R_2$ ......, then the difference of potential at P, Q is (Art. 39)

$$q_1\left(\frac{1}{r_1} - \frac{1}{R_1}\right) + q_2\left(\frac{1}{r_2} - \frac{1}{R_2}\right) + \ldots\ldots\ldots = \Sigma q\left(\frac{1}{r} - \frac{1}{R}\right).$$

If this expression be positive, P has positive potential to Q, or work has to be done to bring a particle from Q to P, and if this expression be negative, P is said to have negative potential to Q, so that work has to be done to bring the particle up from P to Q. When P is positive to Q, if a channel of communication were opened P and Q would instantly be brought to the same potential. This is commonly expressed by saying that + Electricity flows from P to Q. If P were negative to Q the flow of electricity under the same circumstances would be in the opposite direction. This convention is clearly in accordance with our analogy of difference of level, since water would always flow from a place of higher to a place of lower level if a communication were opened for it, quite independently of the absolute level of the two places.

5. *Absolute Potential at a point.* If the point Q is so distant from all parts of the system that every $\frac{1}{R}$ may be

neglected compared with every $\frac{1}{r}$, then we may neglect the $\frac{1}{R}$'s altogether from the expression and we have $\Sigma\frac{q}{r}$ for the absolute potential at a point. Here there is no change of sign as in Art. 39, Cor. 2: since when $\Sigma\frac{q}{r}$ is positive work has to be done on the + unit to bring it from a distant point, or the zero of potential, up to the point indicated in the field.

61. Premising these extended definitions, we proceed to deduce some important results.

Prop. I. The potential over the surface and within the mass of an electrified conductor is constant, and lines of force cut the surface at right angles.

This follows from our first experiment: for since there is no electrical force within the charged conductor there can be no change of potential, or, in other words, the potential is constant. That the potential is constant within the mass of any conductor, might be taken as the definition of a conductor, since if a difference of potential existed between any two points within it, there would be a field of force within the conductor and it would therefore not be a conductor.

Since lines of force cut at right angles every equipotential surface, lines of force emanate at right angles from every conducting surface.

If the potential is constant throughout the mass of a conductor it must also be constant throughout any internal cavities of the conductor (Art. 48, Cor.), since there is no electrical distribution over the internal conducting surface unless other electrified bodies be within the cavity and insulated from it.

Cor. 1. The law of density on a freely electrified conductor is the same as the law of thickness of a film of matter, which exerts no attraction on an internal point.

Cor. 2. Whenever a difference of potential exists between two bodies, which are connected by a conductor, after

some time, short or long, equality of potential is established. This is *said* to take place by a flow of positive electricity from the place of higher to that of lower potential. Its special investigation we defer till we consider electricity in motion.

62. Prop. II. The strength of field or electrical force per + unit just outside a conductor electrified either freely or by induction, at a point where the density is ρ, is $4\pi\rho$.

If we take a small tube of force originating in the surface of the conductor, whose area of section is σ, and the force over which is F, then $F\sigma$ is constant throughout the tube, and on passing through the electrified surface changes by $4\pi\rho\sigma$. But within the conductor F vanishes, and hence just outside the conductor $F\sigma = 4\pi\rho\sigma$ or $F = 4\pi\rho$.

This might also be proved by considering the force within and without, near a small element of the surface, taken near the point under consideration.

Suppose AB such an element, and let P, Q be two points, one just inside, and the other just outside the surface. We may then consider the attraction of AB, and the rest of the conductor ACB, separately from each other.

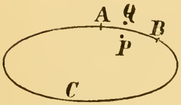

Fig. 15.

It is clear that AB exerts on a + unit* at P a force equal and opposite to that of ACB, and since AB may be treated as a flat plate, uniformly electrified, this force is $2\pi\rho$ inwards.

Again, the force exerted by the part ACB on a + unit at Q is the same as that which it exerts on P, and the force due to AB on a + unit at Q is similarly $2\pi\rho$ outwards.

Hence, the total force on a + unit at Q is $2\pi\rho$ due to AB, and also $2\pi\rho$ due to ACB, both outwards, or the whole force on a + unit just outside the conductor is $4\pi\rho$.

Cor. The force which an electrified conductor exerts on any portion of its electrification is normal, and at the rate of $2\pi\rho^2$ per unit of area. For considering the element AB, the

* It must be understood in this and other cases where we are considering the strength of a field that the + unit is imaginary, for were it placed at the point it would induce a distribution of electricity over the surface in addition to that whose effect we are considering.

force due to ACB on a + unit at any point on AB is $2\pi\rho$, and the quantity of electricity on AB is $\rho\sigma$, if σ be the area of the element AB. Hence the whole force on the electrification of AB is $2\pi\rho^2\sigma$. Hence the force exerted on the electrification of AB is at the rate of $2\pi\rho^2$ per unit area.

63. **Prop. III. If a tube of force cut through two oppositely electrified surfaces the quantities of electricity on its two ends are equal and of opposite sign.**

For supposing F, σ to represent the force per + unit and area of section at one surface, and F', σ' the force per + unit, *measured in same direction,* and section of tube at the other surface,

$$F\sigma = F'\sigma'.$$

But $F = 4\pi\rho$, if ρ be the density on one surface,

$$F' = -4\pi\rho', \text{ if } \rho' \text{ be the density on the other surface};$$

$$\therefore 4\pi\rho\sigma = -4\pi\rho'\sigma';$$

$$\therefore \rho\sigma = -\rho'\sigma',$$

$$\text{or } q = -q',$$

where q, q' are the quantities of electricity on the two surfaces respectively.

64. **Prop. IV. A closed conducting shell screens completely from each other the fields of force due to electrical separations made on opposite sides of it.**

Suppose in the figure we have a hollow conducting body, and let there be within its inner surface def one system of electrified bodies a, b, c, and also without its outer surface DEF another system A, B, C.

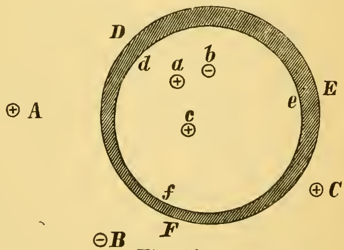

Fig. 16.

The system A, B, C will bring the surface DEF to a constant potential and both the substance of the conductor and the space inside it will be at that same potential. Hence there is no field due to A, B, C within the conductor.

The system a, b, c will bring the surface *def* to a constant potential and the whole conductor and space outside it will as far as they are concerned be at that same potential. Also if the electrical separations be made within *def* the total electrification of a, b, c and the surface *def* will be nil, and there will be no complementary charge on DEF. Hence the field of force due to a, b, c will be enclosed by the surface *def* and will have no existence outside it.

Hence we have two fields of force quite independent of each other; that due to a, b, c inside the hollow conductor and that due to A, B, C outside it.

COR. If the conducting shell be connected with the earth it will be always at zero potential and the two electrical systems a, b, c and A, B, C within and without will be wholly screened from each other whether their total amounts vanish or not.

65. We can now explain why the potential of the earth both for practical and theoretical purposes may be most conveniently chosen as our zero of potential. Since all electrical experiments (except those on atmospheric electricity) are performed in rooms which are simply hollows in a large conductor (the Earth), all our fields of force will (Prop. IV.) consist of the space between our instruments and the walls of the room where the experiments are made, these walls being conducting and being cut at right angles by the lines of force which terminate in them. Hence no work will be done by the electrical system inside the room if the + unit be carried about outside the room, and the work done in carrying it from the walls to a point inside the room is exactly the same as that which would be done in carrying it from infinity to the same point. The expression $\Sigma \frac{q}{r}$ (see Art. 60), supposing the summation to include the complementary distribution on the walls of the room, will be the difference of potential between the walls of the room and the point under consideration. Making the earth our provisional zero we may therefore apply with the highest accuracy all our formulæ derived from the hypothesis of an absolute zero

at an infinite distance from all electrification. Of course external to the room there may be a field due to the earth's electrification, but the last proposition shows that this field is quite independent of that inside the room, only raising or lowering the potential of the room and everything in it by a certain amount, so that as far as the electrical actions inside the room are concerned we may treat our electrical separation as the only one in the universe*.

66. A body will have positive or negative potential according as it is positive or negative relatively to the earth. This must be clearly distinguished from a positive or negative electrification, since a negatively electrified body may have positive potential owing to being placed in a region where there is, owing to other electricity, a numerically higher positive potential than its own negative potential, and *vice versa.* A positively or negatively electrified body is one which when separated to a great distance from all other conductors is respectively positive or negative to the earth.

67. If we have an electric system in a room which is very large compared to the greatest dimensions of the system, we may often neglect entirely for points near the system the action of the complementary charge on the walls of the room, the potential $\Sigma \frac{q}{r}$ only including the charge of the system itself. In this case the body is said to be freely electrified.

* The analogy noted above between potential and temperature is very noticeable in relation to electrostatic potential. Just as temperature at a point expresses a certain condition in relation to heat, so electrostatic potential at a point expresses the condition at that point in relation to electricity. Just as we should pass through our lives without any sensible knowledge of temperature if we always lived in a medium whose temperature was the same or sensibly the same as that of our own bodies, so we do pass through our lives without any sensible knowledge of potential, because we and our surroundings are always at the same potential as the earth. If our bodies are ever in a region where there is a rapid change of potential, we become aware of it by the erection of our hairs, by a tickling sensation on our skin, and by giving off of sparks when we approach bodies at a different potential to our own, and by a shock through our nervous and muscular system (which often results in death) if for an instant different parts of our bodies are at a large difference of potential. It is only the property that all conductors are equipotential throughout that makes the appeal to the senses less understood in electricity than in heat.

68. Prop. V. If one distribution of electricity make the potential V_0 at a given point, and another distribution over the same system make the potential V_1, the potential due to the two distributions will be $V_0 + V_1$.

For if there be quantities of electricity at a point distant r from the given point, q_0 and q_1 corresponding to the potentials V_0 and V_1, the term in the total potential corresponding to these will be $\frac{q_0+q_1}{r}$ or $\frac{q_0}{r}+\frac{q_1}{r}$, and since this will be true for each element, it will be true for the sum, and we shall have the whole potential

$$=\Sigma\frac{q_0+q_1}{r}=\Sigma\frac{q_0}{r}+\Sigma\frac{q_1}{r}=V_0+V_1.$$

Cor. If the potential of a conductor due to a certain quantity Q of electricity be V, the potential due to nQ distributed according to the same law will be nV.

69. Prop. VI. There cannot be two different laws of distribution of electricity on a given conductor.

1. *For a free distribution.* If possible, let there be two such laws, then they must both produce a constant potential within the conductor. Hence, if distributions according to the two laws be superimposed on each other, the combined distribution will produce a constant potential within the conductor. Let now equal amounts of positive and negative electricity be spread over the surface according respectively to the two laws of distribution referred to. At parts they neutralize each other, and at parts there is an excess of positive electricity, in others an excess of negative electricity. Hence a free distribution, partly positive and partly negative, produces a constant potential within the conductor, a result obviously absurd.

2. *For an induced electrification.* If we reverse the sign of electrification in each body, distributing the positive electricity according to one law and the negative according to another, we shall have a system of bodies, the total electrification of each being nil, with electrifications partly positive and partly negative over the surface of each. This is clearly an impossible distribution, since each body would instantly become neutral.

70. We will now investigate the relation between the charge and potential of a conductor. To do this we first define capacity.

DEF. CAPACITY. *The quantity of electricity which will bring a conductor from zero to unit potential, is defined to be the capacity of the conductor.*

It is clear that the capacity of a conductor depends not only on the conductor itself, but on all surrounding electrified and unelectrified bodies.

Prop. VII. If C be the capacity of a conductor removed from all other conductors, which is raised from zero to potential V by a charge Q of electricity, Q = CV.

For since the electrification of the conductor can be only according to one law, it is clear that each increment of the charge is spread over the conductor according to the same law, and the density at each point is altered in the same ratio. Hence the sum $\Sigma \frac{q}{r}$ or the potential will be altered in the same ratio, or the change in V is always proportional to the change in Q. But when $V = 1$, $Q = C$. Hence $Q = CV$ always.

COR. It follows that if a conductor be in a region at potential V_0, and be brought up to potential V, the quantity of the charge is $C(V - V_0)$, since the potential without any free charge is V_0. The potential of a body so electrified, examined by an electrometer entirely in the region at V_0, will be $V - V_0$, but examined by an electrometer with one pole to earth it will be V.

71. Prop. VIII. If a distribution of electricity over a closed surface produce a force at every point of the surface perpendicular to it, this distribution will produce a constant potential at every point within the surface.

Since the resultant force at any point has no component along the surface, the rate of change of potential along the surface vanishes, and the surface is a surface of constant potential.

If the potential at every point within the surface be not the same, draw within it a system of equipotential surfaces

and tubes of force. The equipotential surfaces can in no case cut the surface of the conductor, since the lines of force are everywhere perpendicular to it. Hence, as we proceed inwards, the successive surfaces must constantly diminish in area, and at last vanish. Conceive now any tube of force proceeding from the surface inwards. Throughout it $F\sigma$ is constant, and at some point within the surface σ must vanish, and at this point F must be infinite. But F can only be infinite at a point indefinitely near to another point, having a finite quantity of electricity, and by supposition such a point does not exist within the surface.

Hence we see that the distribution on the conductor is a possible one, and Prop. VI. shows that it is the only one.

72. **Prop. IX. If an equipotential surface belonging to any electrical system be drawn, and a distribution of electricity be made over that surface such that the density at each point is $\frac{\mathbf{F}}{4\pi}$, where F is the resultant force of the system at that point, then this electrification will be in equilibrium and will produce on all external electrified particles the same force as the given electrical system or part of the system enveloped by the surface.**

For let $A, B, C, \ldots$ be a system of electrified particles, and let PQR be an equipotential surface which we will *first* suppose to enclose the system.

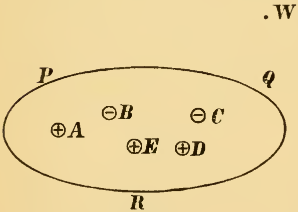

Fig. 17.

Replace for a moment PQR by a conducting surface communicating with the earth which therefore neutralizes any distribution on the outside surface PQR. The distribution on the inner surface PQR will screen an external + unit at W from the action of $A, B, C, \ldots$ (Prop. IV. Cor.). This can only happen through a distribution of electricity on PQR, which produces on W a force exactly equal and opposite to that produced by $A, B, C, \ldots$ (Prop. I. Cor. 1.).

Let now W be close to the surface, say at P. The

resultant force F of A, B, C,... will here be perpendicular to the surface. Hence the resultant force due to the induced charge will also be perpendicular to the surface and will equal $-F$, and this will be true for each point on the surface: and therefore by the last proposition this induced distribution is according to the same law as a free distribution. But for a free distribution of density ρ the force just outside is $4\pi\rho$, hence for the electrical force just inside

$$-F = 4\pi\rho, \text{ or } \rho = -\frac{F}{4\pi}.$$

This gives us the density of the induced distribution.

If we now distribute electricity over PQR, whose density at each point is $+\frac{F}{4\pi}$, we clearly get a distribution which produces on W a force the same in amount and direction as the original distribution A, B, C,... and this distribution is a free distribution.

Hence we may remove the original system A, B, C,... and replace it as far as actions *outside* are concerned by the equipotential surface electrified, so that its electrical density at each point is $\frac{F}{4\pi}$.

Next suppose the equipotential surface to pass between two parts of the system, and let A, B represent the two parts which are external and internal respectively. If the surface be replaced by a conducting surface which is to Earth, there will be two independent distributions, that induced by B on the inside and that induced by A on the outside. Consider the distribution induced by B. It must neutralize the action of B on all external electrified particles. Hence this distribution reversed in sign will give the same strength of field as B at all *external* points. This distribution together with A will therefore give the same strength of field at all external points as the whole system A, B, i.e. it will give a force everywhere perpendicular to the electrified surface. Hence (Art. 71) this distribution is equipotential and (Art. 62) the density

at each point is measured by $\frac{F}{4\pi}$, where F is the resultant force at the point on the surface measured outwards.

COR. 1. The total amount of electricity in the distribution will be equal to that in the part of the system enveloped by the surface.

COR. 2. In the second case since the distribution $\frac{F}{4\pi}$ produces equal potential throughout the conductor, it must be the distribution induced by A on the conducting surface. Hence the distributions due to A's and B's induction are at every point on the surface equal and opposite.

COR. 3. It follows that if we have any non-conducting mass and any system of electrified conductors distributed within it, the resultant force on any external electricity can be represented by a distribution of electricity, partly positive and partly negative, on the bounding surface of the non-conductor. For if we for a moment conceive the bounding surface conducting and connected with the earth, a charge will be induced which screens the electrified bodies inside from all external action. If this charge be reversed in respect of positive and negative and spread over the surface of the non-conductor, this distribution satisfies the condition of the problem.

73. We have shown (Art. 72) that when an equipotential surface passes between two parts A, B of a system so as to envelope B, a distribution of electricity whose law of density is $\frac{F}{4\pi}$ gives at every external point a strength of field equal to that of B. For this reason B is called the electrical image of the electrified surface at all external points. Similarly a distribution whose density is $-\frac{F}{4\pi}$ gives a force equal to A at all points within the surface, and A is therefore the electrical image of this distribution at all internal points.

DEF. ELECTRICAL IMAGES. *An electrical image of a distribution of electricity on a given surface is a point or*

series of points on one side of the electrified surface, which if charged with certain quantities of electricity and substituted for the electrified surface would produce on the other side of that surface the same electrical action as the actual electrification does produce.

If we apply this to a single electrified particle we see that (its equipotential surfaces being spheres) we may substitute for the electrified particle an equal distribution over any spherical surface which has the given particle for centre, and the action of the electrified sphere at all external points will be identical with that of the electrified particle. This gives us an indirect proof of the proposition that the attraction of a freely electrified sphere on any external electricity is the same as if the whole electrification of the sphere were condensed at its centre*.

We defer till next chapter the consideration of some problems in electrical images, a method due to Sir W. Thomson, and one which has in his hands led to the resolution of many problems of the highest order of difficulty. (See papers on Electrostatics and Magnetism.)

74. Prop. X. To determine the law of density over a freely electrified surface.

We have already indicated (Prop. I. Cor. 2) the means by which this can be done, but in almost every particular case the analysis baffles us.

The only method practically useful is indirect and depends on the principle of electrical images. We draw a number of equipotential surfaces for systems of particles with different relative amounts of electricity. From such surfaces we select one which most nearly corresponds to the conductor in ques-

* This also gives an indirect proof of the geometrical theorem that the area of a sphere is 4π times the square on its radius. For let a sphere of radius R be charged with a quantity Q of electricity, the strength of field just outside the sphere is $\frac{Q}{R^2}$ and this equals $4\pi\rho$, where ρ is the density. Hence $\rho = \frac{Q}{4\pi R^2}$: but the density of a uniform distribution is the quotient of the charge by the area. Hence the area is $4\pi R^2$.

tion. Calculating the force at each point on the surface and dividing by 4π, we get the law of density at each point of the free electrification.

We can however lay down one general rule, that the electrical density of a free electrification is always greatest at places of greatest curvature.

If we compare a tube of force starting from a small area at a very convex portion of the surface (Fig. 18) with another tube starting from an equal base on a less convex portion, we see that the more convex the surface, the more rapidly the tube widens out. As we recede from the surface the law of Force approximates to that of inverse square of distances (Art. 56) and all tubes therefore widen as the direct squares of distances, that is, all at the same rate as you retreat from the electrified system. At the same time the equipotential surfaces approximate to spheres and the value of F over each sphere becomes uniform. Thus we see that at great distances, sections of the two tubes remain unequal while the average value of F over each becomes the same. Hence $F\sigma$ must always be greater for the tube starting from the place of high than for that starting from the place of low curvature, or in other words, F is high where the curvature is high. And, since $F = 4\pi\rho$ just outside the surface, where F is high ρ also is high.

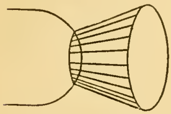

Fig. 18.

It also appears that at an edge or point the density is infinite, for (Fig. 19) we have a tube with a zero base rapidly widening out to a finite area. For $F\sigma$ to be constant throughout such a tube, either F is infinite when $\sigma = 0$, or F vanishes whenever σ is finite, which clearly is not the case. As a matter of fact there is no such thing in nature as a point or an edge, the parts we call such being in reality rounded off. Even if a point did exist we know that the density could not be infinite, since the air would, under high tension, cease to be an insulator, and would conduct away the electricity. This does, in fact, explain the glow always seen in the dark at sharp points when electrified.

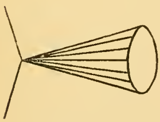

Fig. 19.

75. We have hitherto referred to electrical actions as taking place in air, and assuming that the effects might be represented by action at a distance, have made no reference to the dielectric across which these actions take place. This was the universally accepted view on the subject till Faraday by a series of experiments established the theoretical result that all actions apparently at a distance are the outcome of actions taking place in the intervening dielectric, and also that the nature of the dielectric influences the amount of these actions.

To explain Faraday's Theory of Inductive Action we must conceive the air or other dielectric to consist of a number of conducting molecules, separated from each other by layers of insulating material. We may perhaps represent the medium as consisting of a number of small metallic shot bedded in and kept apart from each other by shellac. If now we conceive a positively electrified conductor surrounded by such a medium, the effect of the electrification is to separate the electricities in the layer next the body, each shot acquiring a positive and a negative pole, the negative pole being directed towards the conductor. This layer of shot produces an exactly similar electrical separation in the layer next to it, and so on through the whole dielectric, the poles of the consecutive molecules always being along the lines of force. The degree of electrical separation in each molecule depends on the amount of the original electrification, to which the whole amount separated over any equipotential surface is equal.

It is easily shown that the amount of electrical separation across any equipotential surface bounded by a given tube of force is measured by $\pm\frac{1}{4\pi}F$ per unit area. For if the surface become conducting and Q be the quantity of electricity on the base of any tube of force originating in the conductor, $-Q$ will be (Art. 63) separated inwards and $+Q$ therefore outwards. But if ρ be the density of the electrification on the base of any tube of area σ, $Q=\rho\sigma=\frac{F\sigma}{4\pi}$ (Art. 62); and since the product $F\sigma$ is constant throughout the tube, F and

σ may be measured for any equipotential surface the tube cuts. Assuming the same separation to take place in the dielectric as in the conducting surface its amount will be $\pm \frac{F\sigma}{4\pi}$ over any small area σ.

We may express this by saying that the quantity separated per unit of area across any equipotential surface is $\pm \frac{F}{4\pi}$, an expression we have already found (Art. 62) for the density of a free electrification.

The medium is by this means put in a state of strain, the lines of strain being the lines of force. The medium when strained tends to return to the normal state by a discharge of electricity from molecule to molecule, and the greater or less facility with which this is effected constitutes better or worse conduction. A good conductor cannot withstand a very small strain, while a good insulator only yields to a very violent strain. All bodies in nature fall between the limits of a perfect conductor and a perfect insulator.

We proceed to consider the effect of changing the dielectric on the electrical actions of a system.

76. Prop. XI. In any given system charged with a given quantity of electricity, the effect of changing the dielectric is to alter the potential of all bodies in the system in a certain ratio.

Take the simplest case, that of a conductor immersed in a medium and freely electrified.

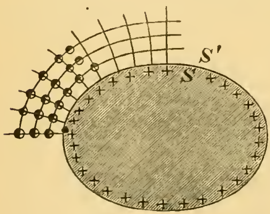

Fig. 20.

Supposing the medium to extend to an indefinite distance round the conductor, the whole effect of the system of electrified molecules composing the medium may be represented by a distribution of electricity over the inner surface of the dielectric (Prop. IX. Cor.).

Again, assuming the dielectric to be electrically homogeneous (isotropic), or to have no electric polarity, we see

that a change in the dielectric cannot produce any change in the form of the system of equipotential surfaces.

The only effect therefore of changing the dielectric is to alter, in a certain ratio, the *effective* electrification at each point. Thus if ρ be the density at any point on an electrified conductor, conceived apart from any medium, the effective density on that point, when immersed in one medium A, becomes $\frac{\rho}{K}$, and when in another medium B, it becomes $\frac{\rho}{K'}$, where K and K' are constants depending only on the media.

Again, since the potential at any point is $\Sigma\frac{q}{r}$, which is the same as $\Sigma\frac{\rho\sigma}{r}$, where ρ is the density on a small area σ, it is clear that the potential V when in medium A becomes $\frac{V}{K}$ and when in medium B, $\frac{V}{K'}$.

This proves the proposition for a conductor freely electrified in space, and the same method of proof can clearly be extended to any system of electrified bodies whatever, since in that case the whole system of equipotential surfaces must remain the same, and that can only occur when the electrification is everywhere altered in the same ratio.

77. If we have the same conductor immersed first in a medium A and then in a medium B and brought from zero up to the same potential V, the quantities of electricity are in the ratio K to K'. For in the medium A the effective quantity is $\frac{Q}{K}$ and therefore $\frac{Q}{K}=CV$ and $Q=CKV$. Similarly in the medium B, $Q'=CK'V$. Hence as $Q:Q'::K:K'$.

The ratio K to K' for the two media A and B is called the ratio of their *specific inductive capacities*. Since we know nothing of the behaviour of a conductor removed from any

medium we can only compare different media. Our standard of reference is air, and its specific inductive capacity is taken as a unit and the capacities of all other media compared with it.

Faraday has shown experimentally that the specific inductive capacity of all gases whatever at all temperatures and pressures is the same, and it is this circumstance, combined with its excellence as an insulator, which makes air so convenient as a standard. He also found that for all the solid and liquid dielectrics he experimented on, the specific inductive capacity was greater than for air.

DEF. SPECIFIC INDUCTIVE CAPACITY *of any dielectric is the ratio of the charge on a conductor immersed in it to the charge on the same conductor raised to the same potential in air.*

In our future investigations we shall, unless the contrary is stated, assume all actions to take place in air, our formulæ then being identical with those proved for action at a distance, and in any case where the dielectric is different from air we shall simply have to multiply the capacity of each conductor by the specific inductive capacity of the dielectric in question.

78. Prop. XII. To calculate the energy exerted in charging any conductor.

By definition the potential is the work done in bringing a + unit of electricity from zero up to the given potential, and if Q units of electricity be brought up from potential zero to potential V, the energy exerted is QV. This however is only true on the supposition that the whole amount of electricity at potential V is so large that the addition of the quantity Q does not sensibly raise the potential. If, however, Q represent the whole charge we should infer that the energy would be $\frac{1}{2}QV$, since at the beginning the potential is at zero at the end at V, and consequently the average potential is $\frac{1}{2}V$ and the whole energy exerted $\frac{1}{2}QV$.

We may show the same result by the graphical method (Art. 6, Fig. 2) representing quantities by abscissæ and potentials by ordinates. Since the rise in potential is in a

constant proportion to rise in quantity the extremities of the ordinates are on a straight line. If we now suppose the charge made by successive small quantities and construct the corresponding parallelograms, it is clear that the area of each parallelogram represents the amount of energy expended in raising the quantity represented by its base to the potential represented by its height. Hence the whole energy is represented by the area of the trapezium which here becomes a triangle whose base is Q, height V and area $\frac{1}{2}QV$.

We give still another proof of an algebraical kind of this very important proposition.

Let the whole charge Q be communicated to a conductor of capacity C by n different charges each equal in amount to q, so that $Q = nq$.

The potential communicated to the conductor by the first charge q is $\frac{q}{C}$. On bringing up the second charge q the work done is $\frac{q^2}{C}$ and the conductor acquires the potential $\frac{2q}{C}$. On bringing up the third charge the potential becomes $\frac{3q}{C}$, and the work done in bringing it up is $\frac{2q^2}{C}$ and so on.

Hence the total energy expended in bringing up successive charges each equal to q is

$$\frac{q^2}{C} + \frac{2q^2}{C} + \ldots\ldots + \frac{(n-1)\,q^2}{C} = \frac{q^2}{C}\,\frac{(n-1)}{2}$$

$$= \frac{1}{2}\,\frac{(nq)^2}{C}\left(1 - \frac{1}{n}\right) = \frac{1}{2}\,\frac{Q^2}{C}\left(1 - \frac{1}{n}\right) = \frac{1}{2}QV\left(1 - \frac{1}{n}\right),$$

Q, V being the final charge and potential respectively. Now if the successive charges be made sufficiently small, and the number of them sufficiently great, $\frac{1}{n}$ may be neglected, and we get as before for the whole energy expended in charging the conductor $\frac{1}{2}QV$.

The principle of the conservation of energy shows us that the energy which runs down in the discharge is equal to the energy which is exerted in the charge, or we may prove it independently by assuming the discharge to take place by a series of n discharges of quantity q.

The sum of the energy which runs down in the successive discharges will be

$$\frac{(Q-q)}{C}q + \frac{Q-2q}{C}q + \dots + \frac{Q-(n-1)q}{C}q$$

$$= \frac{(n-1)q^2}{C} + \frac{(n-2)q^2}{C} + \dots + \frac{q^2}{C},$$

the same series as above.

Hence we see that the energy of charge or discharge is measured by $\frac{1}{2}QV$.

79. If we have any system of conductors charged with given quantities of electricity, the energy expended in charging the whole system is the sum of the energies exerted in charging the separate conductors.

It might at first sight appear that the *order* in which the different conductors of a system are charged would affect the energy, since each charge alters not only the potential of the body in question but inductively that of all other bodies in the system. The conservation of energy shows however that the order of charge or discharge must be on the whole immaterial, as otherwise by continually charging a system in one order and discharging it in a different order there would be a gain of energy. The same principle shows us that if we charge a system of conductors, insulate them and move them about in any way relatively to each other, the whole work done against electrical forces is the excess of the energy after the movements have taken place over the energy of the system when first electrified. We shall illustrate the use of this principle hereafter.

80. Prop. XIII. To investigate the electrification of two conductors subject to each other's inductive influence.

We will call the conductors A_1 and A_2. Let them be

insulated and uncharged at first. Suppose now A_1 charged with a unit of positive electricity, and let its potential be called p_{11}, and let the potential of A_2 by induction of A_1 be p_{12}. We will call p_{11} the coefficient of potential of A_1 on itself and p_{12} the coefficient of potential of A_1 on A_2. If there be a charge Q_1 on A_1 the potential of $A_1 = p_{11}Q_1$ and that of $A_2 = p_{12}Q_1$.

Next let A_2 receive a charge of one unit of positive electricity and let its potential be p_{22} and let the potential of A_1 by induction be p_{21}. Then if A_2 have a charge Q_2 the potential of $A_2 = p_{22}Q_2$ and that of $A_1 = p_{21}Q_2$.

If the two conductors be then at potentials V_1 and V_2 we have by superposition of electrifications (Art. 68),

$$V_1 = p_{11}Q_1 + p_{21}Q_2,$$
$$V_2 = p_{12}Q_1 + p_{22}Q_2,$$

which give the potentials in terms of the quantities of the several electrifications.

We can now solve these equations and obtain the charge of each conductor in terms of the potentials. We have by ordinary algebra

$$Q_1 = \frac{p_{22}}{p_{22} \cdot p_{11} - p_{12} \cdot p_{21}} \,.\, V_1 + \frac{p_{21}}{p_{12} \cdot p_{21} - p_{22} \cdot p_{11}} \,.\, V_2,$$

and

$$Q_2 = \frac{p_{12}}{p_{12} \cdot p_{21} - p_{22} \cdot p_{11}} \,.\, V_1 + \frac{p_{11}}{p_{22} \cdot p_{11} - p_{12} \cdot p_{21}} \,.\, V_2.$$

We will write these equations in the form

$$Q_1 = q_{11} \,.\, V_1 + q_{21} \,.\, V_2,$$
$$Q_2 = q_{12} \,.\, V_1 + q_{22} \,.\, V_2.$$

Here q_{11} is clearly the quantity of electricity which will bring A_1 to unit potential when A_2 is at zero, and may be defined as the coefficient of self-induction of A_1 or (Art. 70) the capacity of A_1, q_{22} being similarly the coefficient of self-induction or the capacity of A_2: q_{12} will be the charge induced on A_2 kept at zero potential when A_1 is at unit potential, and may be called the coefficient of induction of A_1 on A_2, q_{21} being similarly the coefficient of induction of A_2 on A_1.

81. Prop. XIV. To find the energy of the electrification of the system of two conductors.

If we have A_1 charged with a quantity Q_1 and at potential V_1 its energy is $\frac{1}{2}Q_1V_1$. Similarly the energy of A_2 with charge Q_2 and potential V_2 is $\frac{1}{2}Q_2V_2$. Hence the energy expressed in terms of the quantity of charge

$$=\tfrac{1}{2}Q_1(p_{11}Q_1+p_{21}Q_2)+\tfrac{1}{2}Q_2(p_{12}Q_1+p_{22}Q_2)$$
$$=\tfrac{1}{2}p_{11}Q_1^2+\tfrac{1}{2}(p_{21}+p_{12})Q_1Q_2+\tfrac{1}{2}p_{22}Q_2^2.$$

Similarly if expressed in terms of the potential the energy

$$=\tfrac{1}{2}q_{11}V_1^2+\tfrac{1}{2}(q_{21}+q_{12})V_1V_2+\tfrac{1}{2}q_{22}V_2^2.$$

82. Prop. XV. To prove that $\mathbf{p_{12}=p_{21}}$ and $\mathbf{q_{12}=q_{21}}$.

The first of these easily follows by considering the energy added to the system by bringing up a small addition x_1 to the charge of A_1.

Using the formula of preceding proposition the energy after the addition of x_1 to the charge

$$=\tfrac{1}{2}p_{11}(Q_1+x_1)^2+\tfrac{1}{2}(p_{12}+p_{21})(Q_1+x_1)(Q_2+\tfrac{1}{2}p_{22}Q_2^2.$$

Expanding and neglecting x_1^2 and subtracting the original energy we have the increment in energy

$$=\{p_{11}Q_1+\tfrac{1}{2}(p_{12}+p_{21})Q_2\}x_1.$$

But this must equal the work done in bringing x_1 up to A_1 at potential V_1

$$=V_1x_1$$
$$=\{p_{11}Q_1+p_{21}Q_2\}x_1.$$

On comparing these identical expressions for the increase of energy we see

$$p_{21}=\tfrac{1}{2}(p_{12}+p_{21});$$
$$\therefore\ p_{21}=p_{12}.$$

And on comparing the values of q_{12} and q_{21} in terms of p's it is at once seen that

$$q_{21}=q_{12}.$$

Interpreting these results we see that for any two conductors A_1, A_2: (1) the potential of A_1 due to unit charge

on A_2 equals the potential of A_2 due to unit charge on A_1: (2) the charge induced in A_2 kept at zero, by A_1 at unit potential equals the charge induced in A_1 kept at zero by A_2 at unit potential.

These reciprocal theorems in electrification are due to Sir W. Thomson and Helmholtz.

Although we are unable without higher mathematical analysis to express the values of these coefficients except in the simplest cases, we can prove one or two properties as to their relative values.

83. **Prop. XVI. To prove that p_{12} is positive but less than either p_{11} or p_{22}.**

For let A_1 be charged with a unit of electricity, A_2 being uncharged. The potential of A_1 is by definition p_{11} and that of A_2 is p_{12}. But since lines of force proceed from A_1 to all points of space some fall on A_2, and since along a line of force potential constantly decreases, the potential of A_2 is less than that of A_1 or $p_{12} < p_{11}$, and because lines of force proceed from A_2 into space, p_{12} or the potential of A_2 is positive. Similarly p_{21} (which equals p_{12}) $< p_{22}$.

84. **Prop. XVII. To prove that q_{11} and q_{22} are positive but that q_{12} is always negative, and that numerically q_{12} is less than q_{11} or q_{22}.**

Observing the values of q's in terms of p's in Prop. XIII. we see that the signs of q_{11}, q_{22} are determined by the sign of $p_{11}p_{22}-(p_{12})^2$, while that of q_{12} is determined by that of $(p_{12})^2-p_{11}p_{22}$; and by last proposition the former of these is necessarily positive and the latter necessarily negative. This result might have been predicted from the general principle of induction.

It will also be noticed that q_{11} and q_{22} have for their numerators p_{22} and p_{11} respectively, while q_{12} has p_{12} for its numerator, the denominators being numerically the same throughout. Hence in virtue of the relation proved in last Article q_{12} is less numerically than q_{11} or q_{22}.

The method above adopted can clearly be extended to any system of conductors whatever. For this the student is referred to Prof. Clerk Maxwell's *Electricity and Magnetism*, Chap. III., Art. 85, &c.

85. Prop. XVIII. To investigate the electrification of a condenser consisting of two surfaces, one entirely surrounding the other, and brought to different potentials.

This is a particular case of the general problem of two conductors solved in the last proposition.

Suppose A_1 entirely to surround A_2 and apply the formulæ

$$Q_1 = q_{11} V_1 + q_{21} V_2,$$

$$Q_2 = q_{12} V_1 + q_{22} V_2.$$

The following are the interpretations and relations of the coefficients of induction in this case:

q_{22} = charge on A_2 at unit potential when A_1 is at zero potential $= C$, which is called the capacity of the condenser.

$q_{12} = q_{21}$ = charge induced on A_1 when A_2 is at unit potential. This will (Art. 63) $= -q_{22}$ or $-C$ since all the lines of force from A_2 intersect A_1.

q_{11} = charge on A_1 at unit potential when A_2 is at zero potential. This would be equal to q_{22}, but for the induction from A_1 outwards. This induction outwards will be equal to the capacity of A_1 for a free charge. Hence

$$q_{11} = q_{22} + C' = C + C'.$$

Substituting

$$Q_1 = C(V_1 - V_2) + C' V_1,$$

$$Q_2 = -C(V_1 - V_2).$$

The bound charge is therefore $C(V_1 - V_2)$ and the free charge $C' V_1$.

The energy of the electrification

$$\begin{aligned}&= \tfrac{1}{2} Q_1 V_1 + \tfrac{1}{2} Q_2 V_2 \\ &= \tfrac{1}{2} C V_1^2 - C V_1 V_2 + C V_2^2 + \tfrac{1}{2} C' V_1^2 \\ &= \tfrac{1}{2} C (V_1 - V_2)^2 + \tfrac{1}{2} C' V_1^2.\end{aligned}$$

These two terms are respectively the energy of the bound and free charge.

To find the force between the two sheets of the condenser, we must first find the work done in making any small movement. During any movement V_1, V_2 and C' will be unaltered, if we suppose the surfaces kept at fixed potentials. The only variable will be C the capacity of the bound charge. Hence the work done in moving A_2 inside A_1 from a position in which the capacity is C to another in which it is C_1

$$\begin{aligned}&= \tfrac{1}{2} (C - C_1)(V_1 - V_2)^2 \\ &= F \times x,\end{aligned}$$

if F be the force against which the movement is made and x the distance through which A_2 is moved;

$$\therefore \quad F = \tfrac{1}{2} \frac{C - C_1}{x} \cdot (V_1 - V_2)^2.$$

The term $\dfrac{C - C_1}{x}$ depends purely on the geometry of the conductors and is the rate of change of the capacity as the relative movement of the sheets is made.

Thus the force helping or resisting movement in any direction is half the product of the rate of change of capacity in making the movement and the square of the potential difference.

CHAPTER IV.

PROBLEMS IN STATICAL ELECTRICITY.

86. WE have in the preceding chapter given demonstrations of the most important theorems on which the science of Statical Electricity rests, and we now append a series of problems, many of which are of the greatest importance to the practical electriciau, while others are introduced with a view of suggesting to the student methods by which other similar problems may be successfully attacked.

87. Prop. I. To find the potential at any point within a sphere freely electrified with a known quantity of electricity.

Let R be the radius of the sphere, and Q the quantity of electricity. Since the electrification is free, we may neglect all terms depending on the complementary distribution.

Since all parts of the electrification are at the same distance R from the centre, the potential $\Sigma \frac{q}{r}$ becomes at the centre $\frac{\Sigma q}{R} = \frac{Q}{R}$. But the potential at the centre is the same as that throughout the sphere.

Hence if V be the potential at any point within the sphere $V = \frac{Q}{R}$.

To find the capacity of the sphere, we have only to remember that if $V = 1$, $Q = C$;

$$\therefore \; 1 = \frac{C}{R} \text{ or } C = R.$$

Hence the capacity is numerically equal to the radius.

We might therefore define unit capacity as the capacity of a sphere whose radius is one centimetre electrified freely to unit potential.

88. We now give a direct proof of the above proposition. Let APB be a spherical shell freely electrified with a charge whose density at any point is ρ.

Let O be any point inside the sphere, and AOB the diameter. We may conceive the sphere as generated by the revolution of a circle APB round the diameter AB.

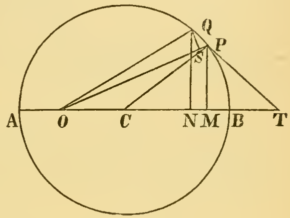

Fig. 21.

If we take two points PQ very near each other on the circle, and draw perpendiculars PM, QN to the diameter, it is clear that PQ by its revolution traces out an annulus, whose radius is PM, and breadth PQ, every point on which will be assumed equidistant from O.

Join OP, OQ, CP; draw QS perpendicular to OP, and join PQ, producing it to cut AB produced in T.

Area of the annulus traced by the revolution of PQ about $AB = 2\pi PM \,.\, PQ$;

$$\therefore \text{ potential at } O \text{ due to the annulus} = 2\pi\rho \,.\, \frac{PM \,.\, PQ}{OP}$$

$$= 2\pi\rho \,.\, PQ \sin POC = 2\pi\rho \,.\, \frac{PS}{\cos QPO} \,.\, \sin POC$$

$$= 2\pi\rho \,.\, PS \,.\, \frac{\sin POC}{\sin OPC} = 2\pi\rho \,.\, \frac{a}{f} PS,$$

if a = radius of sphere, and $f = OC$.

Now since OQS is a right-angled triangle, whose vertical angle is exceedingly small, no appreciable error will be committed if we assume $OQ = OS$, and make $PS = OP - OQ$.

Hence the potential at O due to the annulus

$$= 2\pi\rho \frac{a}{f} (OP - OQ).$$

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If we add these successive differences for all the annuli of which we may suppose the sphere composed, we have for the potential at O due to the whole shell

$$V = 2\pi\rho \cdot \frac{a}{f} \cdot (OB - OA)$$

$$= \frac{2\pi\rho a}{f}(\overline{a+f} - \overline{a-f})$$

$$= 4\pi\rho a,$$

which is independent of f, and therefore constant for all internal points.

89. We can now deduce the area of the sphere by summing the areas of the elementary annuli. The area of the annulus formed by the revolution of PQ

$$= 2\pi PM \,.\, PQ$$

$$= 2\pi OP \,.\, \sin POC \,.\, PQ$$

$$= 2\pi OP \,.\, \sin POC \,.\, \frac{PS}{\cos QPO}$$

$$= 2\pi \frac{a}{f} OP\,(OP - OQ), \text{ by Art. 88.}$$

We may assume without error $2OP = OP + OQ$;

$$\therefore \text{ area of annulus} = \pi \frac{a}{f}(OP^2 - OQ^2).$$

Adding up all the annuli, the whole area

$$= \frac{\pi a}{f}(OB^2 - OA^2)$$

$$= \frac{\pi a}{f}(\overline{a+f}|^2 - \overline{a-f}|^2) = 4\pi a^2.$$

Hence if the electrical distribution have a uniform density ρ, the whole quantity

$$Q = 4\pi a^2 \rho;$$

$$\therefore V = 4\pi a\rho = \frac{Q}{a},$$

which agrees with the former result.

90. Prop. II. To show that the potential due to a uniformly electrified spherical shell at any point without it, is the same as if the whole quantity were collected at its centre.

Making the same construction as before, remembering that O is now external, we have by Art. 88,

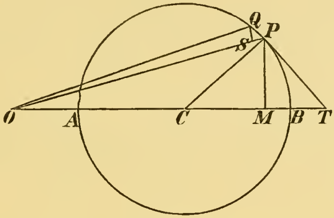

Fig. 22.

$$\text{potential at } O \text{ due to the annulus} = \frac{2\pi\rho PM \,.\, PQ}{OP}$$

$$= 2\pi\rho \,.\, PQ \sin POC$$

$$= 2\pi\rho \,.\, \frac{PS}{\cos OPQ} \,.\, \sin POC$$

$$= 2\pi\rho PS \,.\, \frac{\sin POC}{\sin OPC} = 2\pi\rho \frac{a}{f} \,.\, PS$$

$$= \frac{2\pi\rho a}{f}(OP - OQ).$$

Summing up the successive differences, we have

$$\text{potential at } O \text{ due to the sphere} = \frac{2\pi\rho a}{f}(OB - OA)$$

$$= \frac{2\pi\rho a}{f}(\overline{f+a} - \overline{f-a})$$

$$= \frac{4\pi\rho a^2}{f} = \frac{Q}{f},$$

remembering that $Q = 4\pi a^2 \rho$.

Hence the potential due to the shell is the same as if the whole quantity were collected at its centre.

Cor. It follows that the attraction of a uniformly electrified spherical shell on any external electricity is the same as if the whole quantity on the sphere were collected at its centre. For since the potentials are the same at every external point, the rate of change of potential in any direction must also be the same, and this measures the electrical force, which is consequently the same as if the whole quantity of electricity were accumulated at the centre.

91. Prop. III. The average potential over any sphere in space is the same as the potential at its centre, supposing all electricity external to the sphere. (Gauss.)

By the term 'average potential,' we understand that the sphere's surface is cut up into a large number of equal areas; the average potentials over all the areas added, and the result divided by the number of the areas. If the areas are not equal, we must multiply each potential by the area over which it is calculated, and divide by the sum of all the areas. We adopt the latter method, and with our usual notation we define the average potential over the sphere by $\frac{\Sigma V\sigma}{\Sigma\sigma}$, where V is the potential at any point, and σ the elementary area over which V is taken. Consider one electrified particle and let its quantity of electricity be denoted by m.

Then, in Fig. 22, the potential over the annulus PQ whose area is $2\pi PQ\,.\,PM$, due to a quantity m at the point O, may be taken as $\frac{m}{OP}$;

$$\therefore V\sigma = \frac{2\pi PM\,.\,PQ\,.\,m}{OP}$$
$$= 2\pi m\,.\,\frac{PM\,.\,PQ}{OP}.$$

And by last Article,

$2\pi m\,.\,\frac{PM\,.\,PQ}{OP}$ = potential at O due to a distribution of density m over the annulus.

Hence summing over the whole sphere

$$\Sigma\sigma V = 2\pi m \Sigma \frac{PM \,.\, PQ}{OP}$$

$$= \frac{4\pi a^2 m}{f} \text{ (Prop. II.)},$$

and $$\Sigma\sigma = 4\pi a^2;$$

$$\therefore \frac{\Sigma V\sigma}{\Sigma\sigma} = \frac{m}{f} = \text{potential at centre due to quantity } m \text{ at } O,$$

which proves the proposition as far as a single electrified particle is concerned. In the same way the proposition will be true for any system of electrified particles taken separately, and therefore when added together.

92. **Prop. IV. The potential anywhere within an unelectrified conducting sphere is the same as the potential at its centre due to the inducing electrical system.**

The potential at the centre is made up of the potential of the inducing system, and of the induced distribution on the sphere. But since the sphere's electrification is only induced, there must be equal amounts of positive and negative electricity equally distant from the centre. Hence the potential due to the induced charge is nil, and the only potential at the centre is that due to the inducing system. But by Art. 61, in every case the potential throughout the sphere is the same as at its centre.

Cor. 1. If the sphere were first raised to a given potential, and then introduced into the electrical system, the potential of the sphere would be raised by the potential at its centre due to the electrical system.

Cor. 2. It also follows that if a sphere of radius a be charged with a quantity Q, and placed near a system of external electrified particles, containing quantities $m_1, m_2 \ldots$ of electricity, and at distances $f_1, f_2 \ldots$ from the centre of the sphere, the potential will be given by

$$V = \frac{Q}{a} + \Sigma \frac{m}{f};$$

and supposing the sphere brought to potential V, the quantity of its electrification is given by

$$Q = a \,.\, V - a\Sigma \frac{m}{f}.$$

Cor. 3. This principle may be extended to any conductor. Let A be the charged body, and B an unelectrified body near it. The force due to B's electrification at any point within it, is equal and opposite to the force due to that of A. Hence the equipotential surfaces due to B's electrification and to A's, coincide in position, but are not of the same absolute value. Again, considering B's electrification alone, its potential would clearly be negative near b and positive near c. We see therefore that the potential of B is intermediate between the potentials at b and c due to A, and the function of the induced negative charge at b is to keep the potential down, and that of the positive charge at c to keep the potential up to the mean value.

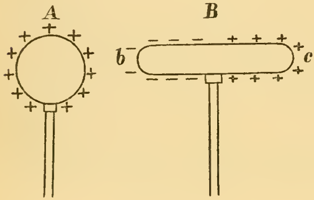

Fig. 23.

We cannot assume however that the potential of B will be negative or positive as its electrification is negative or positive, so that the surface of zero potential of B's electrification passes through the line of neutral electrification.

93. Prop. V. To investigate the potential of a system consisting of a sphere and a concentric spherical shell insulated from it, both being charged with known quantities of electricity.

Let O be the common centre, A the sphere charged with a quantity Q of electricity, B the inner and C the outer surface of the spherical shell, which is charged with Q' units of electricity.

Now we have to consider not only the distribution of Q on A, and Q' on C, but also the charge induced on B, which will be, by Art. 64, equal and opposite to Q (i.e. $= -Q$), the distribution complementary to this going to the outer surface C and making a quantity $Q+Q'$ on C.

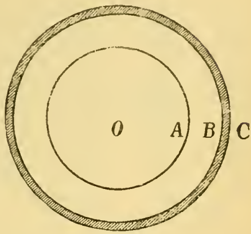

Fig. 24.

Then potential throughout A is same as potential at O, and

$$= \frac{Q}{OA} - \frac{Q}{OB} + \frac{Q+Q'}{OC}.$$

The potential just outside C is the same as if all the electricity were collected at O and $\therefore = \frac{Q+Q'}{OC} = V_C$ suppose.

If V_A be the potential within A, $V_A - V_C$ is the difference of potential between the two coats of the condenser formed by A and B.

Hence $V_A - V_C = \frac{Q}{OA} - \frac{Q}{OB} = \frac{Q}{R} - \frac{Q}{R'}$, suppose.

Again, if $V_A - V_C = 1$, $Q = C$, the capacity of the condenser for a bound charge

$$\therefore C = \frac{RR'}{R'-R}.$$

Thus we see that if $R'-R$ be sufficiently diminished the capacity becomes as great as we please, and the arrangement is therefore called a condenser.

There is in addition a free charge on the outer coat which has for its capacity the radius OC.

94. Prop. VI. To find the capacity of a condenser consisting of two parallel plates electrified to given potentials.

Let A, B be the two plates, of which A is at potential V_1, and B at potential V_2.

Neglecting a portion of the plates near the edge, we see that we have *three* electrical systems to consider, the outer surfaces S_1' and S_2' of A and B being freely electrified, while the inner surfaces act on each other; these systems being screened from each other by the substance of the conducting plates. We at present consider the bound charge only, produced by the action on each other of the surfaces S_1 and S_2, which are at potentials V_1 and V_2.

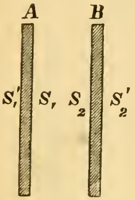

Fig. 25.

Neglecting a portion round the edge, the lines of force which cut both surfaces at right angles are a system of parallel lines: the tubes of force formed by them are cylinders: and in virtue of the relation $F\sigma =$ constant, and $\sigma =$ a constant, F must be constant everywhere between A and B.

Again, the potential between A and B changes from V_1 to V_2.

Hence the rate of change of potential is $\dfrac{V_1 - V_2}{t}$, where t is the distance between S_1 and S_2, and this represents the strength of field at any point between these surfaces.

Again, if ρ be the density at any point on an electrified surface, the strength of field just outside it $= 4\pi\rho$;

$$\therefore \frac{V_1 - V_2}{t} = 4\pi\rho,$$

$$\text{or } \rho = \frac{V_1 - V_2}{4\pi t};$$

and if S be the area of the surface, $Q = \rho S$;

$$\therefore Q = \frac{(V_1 - V_2) S}{4\pi t}.$$

But if $\quad V_1 - V_2 = 1,\ Q = C$, the capacity;

$$\therefore C = \frac{S}{4\pi t}.$$

Again, since tubes of force pass from S_1 to S_2 there must

be equal quantities of electricity, of opposite signs, on the two surfaces.

Hence the quantity on $B = -Q = -\dfrac{(V_1 - V_2)S}{4\pi t}$.

If the dielectric between S_1 and S_2 be not air but some substance whose inductive capacity is represented by K,

$$C = \frac{KS}{4\pi t} = \frac{S}{4\pi \frac{t}{K}} = \frac{S}{4\pi t'}, \quad \text{(Art. 77,)}$$

where $t' = \dfrac{t}{K}$ = thickness 'reduced to air.'

To complete the investigation we ought to find the amount of the free charge on the two surfaces S_1' and S_2'; this can only be done in a few particular cases. If however their capacities be C_1 and C_2, the quantities of the free charges will be C_1V_1 and C_2V_2 respectively.

The same theory can be applied to every form of condenser, provided the thickness be small and the two surfaces everywhere parallel.

In the common form of Leyden jar, where the outer coat is connected with the earth, and the inner coat is nearly a closed surface, the free charge is only the charge of the knob and wire, which project from the inner coat and are used for charging it.

95. Prop. VII. To find the attraction between the opposite plates of the condenser in the last Article.

Let A be a moveable and B a fixed plate.

Then if ρ be the density at any point on A,

$$\frac{V_1 - V_2}{t} = 4\pi\rho;$$

and since the density on B is $-\rho$, the strength of field due to B near an element of A is $2\pi\rho$ (Art. 36); assuming the diameter of B very large compared with the distance between the plates.

If S be the area of A, the quantity of A's electrification is ρS and the force on A is $2\pi\rho \times \rho S = 2\pi\rho^2 S$

$$= 2\pi S\left(\frac{V_1 - V_2}{4\pi t}\right)^2 = \frac{S}{8\pi t^2}(V_1 - V_2)^2.$$

Cor. The difference of potential deduced from an observed force F is given by

$$V_1 - V_2 = t\sqrt{\frac{8\pi F}{S}},$$

the formula employed in absolute and attracted disc-electrometers.

Note. We have already (Art. 85) shown that the force between the two plates of any condenser per unit difference of potential is half the rate of change of capacity. It can easily be seen that $\frac{S}{8\pi t^2}$ is half the rate of change of the capacity $\left(\frac{S}{4\pi t}\right)$, as t the distance is varied.

96. Prop. VIII. Two fixed plates are kept at potentials $\mathbf{V_1}$ and $\mathbf{V_3}$ and a third moveable plate kept at potential $\mathbf{V_2}$ is placed symmetrically between them. To find the resultant force on the middle plate.

Supposing V_2 to be greater than V_1 and V_3, and $V_1 > V_3$.

By the last proposition for the attraction of A on B,

$$F_1 = \frac{S}{8\pi t^2}(V_2 - V_1)^2, \text{ where } S = \text{area of } B:$$

also the attraction of C on B,

$$F_2 = \frac{S}{8\pi t^2}(V_2 - V_3)^2.$$

Hence resultant force on B towards C, the plate of lower potential,

$$= F_2 - F_1 = \frac{S}{8\pi t^2}\{(V_2 - V_3)^2 - (V_2 - V_1)^2\}$$

$$= \frac{S}{4\pi t^2}\left(V_2 - \frac{V_1 + V_3}{2}\right)(V_1 - V_3).$$

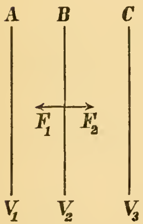

Fig. 26.

The reasoning of Art. 85 proves that if A, C be equal and symmetrically placed with regard to B, a similar formula can be employed. For moving B to one side or the other of the position of perfect symmetry the rates of change of capacity of the systems A, B, and B, C will be equal but of opposite sign. If this rate of variation of capacity be denoted by 2κ, $F_1 = \kappa(V_2 - V_1)^2$, $F_2 = \kappa(V_2 - V_3)^2$, and hence

$$\text{the resultant force} = 2\kappa(V_1 - V_3)\left(V_2 - \frac{V_1 + V_3}{2}\right)$$

$$= 2\kappa V_2 (V_1 - V_3)$$

if the potential of V_2 be very high compared with V_1 and V_3. The conditions of this problem are satisfied by the quadrant electrometer in which the needle B hangs symmetrically between A and C which are in the shape of two hollow quadrants of a circle separated by a narrow air space.

97. Prop. IX. To calculate the energy of the discharge of a Leyden jar or condenser.

This might be written down from the expression for the energy of a condenser (Art. 85). We give now a separate investigation.

Referring to Prop. VI. we see that we have a quantity $\frac{(V_1 - V_2)S}{4\pi t}$ at potential V_1, and a quantity $-\frac{(V_1 - V_2)S}{4\pi t}$ at potential V_2. Hence (Art. 78) the whole energy of the bound charge

$$= \tfrac{1}{2}\frac{(V_1 - V_2)S}{4\pi t}V_1 - \tfrac{1}{2}\frac{(V_1 - V_2)S}{4\pi t}V_2$$

$$= \tfrac{1}{2}\frac{(V_1 - V_2)^2 S}{4\pi t} = \tfrac{1}{2}Q(V_1 - V_2).$$

To this must be added the energy of the free charge which consists of two coats of capacities C_1, C_2 suppose, at potentials V_1 and V_2. The energy therefore is $\tfrac{1}{2}C_1V_1^2 + \tfrac{1}{2}C_2V_2^2$.

This formula also supplies us with the heating power of

the discharge, since when work is done in no other form by an electrical discharge its energy is converted into heat. To give the result in absolute thermal units we must divide by Joule's mechanical equivalent of heat.

98. The whole energy expressed in the above formula can never be obtained in practice when the dielectric is different from air, owing to apparent absorption by the dielectric of part of the charge. Although this portion of the charge can be regained as the residual charge it is obvious there will be a loss of energy when the discharge takes place in two portions instead of all at once. In fact, if V be the potential and Q the quantity of the first discharge, and v the potential and q the quantity of the residual discharge, the energy obtained on the double discharge $= \frac{1}{2}(VQ + vq)$, while the whole energy is $\frac{1}{2}(V+v)(Q+q)$.

99. Prop. X. A Leyden jar having capacity for bound charge C has an inner coat whose free capacity is C_1 and an outer coat whose free capacity is C_2. The jar is charged to potential V and insulated and the knob is then connected with the ground. To find the potential of the outer coat and the charge of the jar.

Let the quantity of electricity on inner coat

$$= Q_1 = (C + C_1)V,$$

and the quantity on outer coat $= Q_2 = -CV$.

When the inner coat is connected with the ground the charge of the outer coat is divided between bound and free charge in the ratio $C : C_2$.

$\therefore$ Bound charge of outer coat

$$= \frac{C}{C+C_2} Q_2 = -\frac{C^2}{(C+C_1)(C+C_2)} Q_1.$$

Free charge of outer coat $= \dfrac{C_2}{C+C_2} Q_2$;

$\therefore$ Potential of outer coat $= \dfrac{Q_2}{C+C_2} = -\dfrac{CV}{C+C_2}$.

Also the amount of electricity on inner coat

$$= \frac{C^2}{(C+C_1)(C+C_2)} Q_1;$$

$$\therefore \text{ Loss} = \left\{1 - \frac{C^2}{(C+C_1)(C+C_2)}\right\} Q_1.$$

100. **Prop. XI. A Leyden jar is charged and insulated. Successive contacts are made with the inner and outer coats. Find the amount of electricity removed by n contacts with inner or outer coat.**

Using the notation of the preceding Article, and writing $\dfrac{C}{C+C_1} = m$ and $\dfrac{C}{C+C_2} = m'$, we see that when the outer coat is to earth, the charge on inner coat is divided in the ratio $m : 1 - m$ between bound and free charge. Also when the knob is to earth the charge on outer coat is divided in the ratio $m' : 1 - m'$. We see therefore

At *first contact with knob:*

Free charge on outer coat $= (1 - m') Q_2$,
Bound charge.............. $= m'Q_2 = - mm'Q_1$.

At *first contact with outer coat:*

Free charge on inner coat $= (1 - m) mm'Q_1$,
Bound charge.............. $= m^2m'Q_1 = - mm'Q_2$.

At *second contact with knob:*

Free charge on outer coat $= (1 - m') mm'Q_2$,
Bound........................ $mm'^2 Q_1 = - m^2m'^2 Q_2$.

By similar reasoning *after* n *contacts with knob:*

Free charge on outer coat $= (1 - m') m^{n-1}m'^{n-1} Q_2$,
Bound........................ $= m^{n-1}m'^n Q_2 = - m^n m'^n Q_1$.

Hence the amount removed by n contacts with knob

$$= Q_1 (1 - m^n m'^n),$$

the quantity removed by n contacts with outer coat

$$= Q_2 (1 - m^n m'^n).$$

But generally C_1 and C_2 are very small compared with C, and therefore m, m' are fractions very near unity. Hence we

see that a large fraction of the charge remains after numerous contacts, and it would require an infinite number of contacts to discharge the jar.

101. Prop. XII. Two Leyden jars are charged to different potentials and afterwards have their knobs brought into contact, the outer coats being kept in connection with the earth. To find the potential of each jar after contact.

Let C_1, C_2 be the capacities, and V_1, V_2 the potentials of the jars, and let V be their common potential after contact. Then since the whole amount on the inner coats is unaltered

$$(C_1 + C_2)\,V = C_1V_1 + C_2V_2;$$

$$\therefore\ V = \frac{C_1V_1 + C_2V_2}{C_1 + C_2},$$

an equation for V.

Cor. It follows that there will always be a loss of energy when two jars at different potentials are united.

For energy before contact $= \frac{1}{2}(C_1V_1^2 + C_2V_2^2)$,

and energy after contact $= \frac{1}{2}(C_1 + C_2)\left(\dfrac{C_1V_1 + C_2V_2}{C_1 + C_2}\right)^2$.

Now $(C_1V_1^2 + C_2V_2^2)(C_1 + C_2) >$ or $< (C_1V_1 + C_2V_2)^2$,

as $$C_1C_2(V_1^2 + V_2^2) > \text{ or } < 2C_1C_2V_1V_2,$$

or as $$(V_1 - V_2)^2 > \text{ or } < 0.$$

The left-hand side is obviously the greater, and hence the sum of the energies of the separate jars is greater than that of the two combined.

102. It was by a particular experimental application of the above that Faraday determined the specific inductive capacity of different substances. He constructed two exactly similar Leyden jars, the coatings of which were so arranged that the dielectric could be changed at pleasure. One of the jars had air for its dielectric, and the other the substance to be experimented upon.

Let now K be the unknown specific inductive capacity; then if C be the capacity of the jar with air, CK is the capacity with the other substance as dielectric.

Let now the jar with air be raised to potential V. On dividing the charge with the other jar, which is uncharged, the potential V' evidently becomes

$$V' = \frac{CV}{C(1+K)} = \frac{V}{1+K};$$

$$\therefore\ V'(1+K) = V,$$

or

$$K = \frac{V-V'}{V'}.$$

Now V and V' are determined by experiment, after the operations indicated, and the value of K for the substance in question becomes known.

103. Prop. XIII. To show that the whole charge in a battery of similar jars charged by cascade only equals the charge of a single jar.

Let A, B, C be such a series of jars of which the first is brought up to potential V, and the last is to earth.

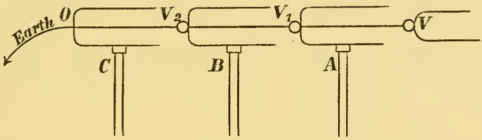

Fig. 27.

Let the potential of the knob of B and of the outer coating of A be V_1.

Let the potential of the knob of C and of the outer coating of B be V_2.

And let C be the capacity of each jar.

Then the *equal* quantities in the three jars are

$$C(V-V_1),\ C(V_1-V_2),\ CV_2.$$

Hence the whole charge in the battery

$$= C(V-V_1) + C(V_1-V_2) + CV_2$$

$$= CV$$

= charge of single jar charged alone to the same potential.

The same method applies to any number of jars.

104. Prop. XIV. To find the work done by a conducting plate communicating with the earth which is allowed to move up parallel to another equal plate which is kept at constant potential.

The principle of Art. 78 might here be employed, since it is clear the work done is stored up in the form of electrical energy, and the energy obtained on discharging the condenser is the equivalent of the work so stored up.

In this case we give an independent investigation.

Let A be the plate kept at constant potential, and B that connected with the earth.

Fig. 28.

By Prop. VII. the force on $B = \dfrac{V^2 S}{8\pi OP^2}$.

The average force through the element PQ is $\dfrac{V^2 S}{8\pi . OP . OQ}$.

Hence the work done by the plate in coming up from P to Q

$$= \frac{V^2 S . PQ}{8\pi . OP . OQ}$$

$$= \frac{V^2 S}{8\pi} \cdot \frac{OP - OQ}{OP . OQ} = \frac{V^2 S}{8\pi}\left(\frac{1}{OQ} - \frac{1}{OP}\right).$$

Adding up the work done on successive elements of the path we see that if t be the ultimate distance,

$$\text{whole work} = \frac{V^2 S}{8\pi} \cdot \frac{1}{t} = \tfrac{1}{2} QV,$$

as might have been anticipated.

105. We now give an example in which the work done is deduced from the change in energy of the system.

Before proceeding to the particular problem it may be well to point out that an electrified body moving through space under electrical forces has no kinetic energy in virtue of its movement, since kinetic energy is proportional to the mass moved, and the electricity has itself no mass. When

work is done on an electrical system, though none of it is converted into kinetic energy, the potential energy of the electrical system is increased by an amount equal to the work done, owing to a redistribution of the electricity in the system.

Prop. XV. Two plates are placed parallel to each other, charged as a condenser, insulated, and separated to an infinite distance. To find the work done in the removal.

Let C be the capacity of the bound charge, and C' the capacity of free charge on the condenser. The free capacity varies with every new position of the plates; we shall however assume here that when both sides of the plates are electrified freely the capacity of the free charge becomes doubled, i.e. when the plates are entirely removed from each other's influence the capacity of each becomes $2C'$.

This assumption would be correct supposing the two plates to begin with were indefinitely near together, since in that case their external surfaces would be electrified as a single plate; but after separation they would be electrified as two separate plates, each of the same size as the former. We can therefore only assume the result as more than approximately true in practice when the distance between the plates is very small.

After charging, the amount on the positive plate is

$$(C+C')\,V,$$

and its energy of discharge is $\frac{1}{2}V^2(C+C')$.

The quantity on the negative plate is $-CV$, and its potential zero.

Hence the energy of the whole system $=\frac{1}{2}V^2(C+C')$.

After removal the quantity on each plate is unaltered, but the capacity, as assumed above, is $2C'$.

Hence potential of positive plate $=\dfrac{C+C'}{2C'}V$,

and the energy of its discharge $=\dfrac{(C+C')^2V^2}{4C'}$;

and the energy of negative plate $=\dfrac{C^2V^2}{4C'}$.

Hence the gain of energy

$$=\frac{(C+C')^2V^2+C^2V^2}{4C'}-\frac{(C+C')V^2}{2}$$

$$=\frac{V^2}{4C'}\{(C+C')^2+C^2-2C'(C+C')\}$$

$$=\frac{V^2}{4C'}(2C^2-C'^2)$$

= work done in separation of plates.

106. The next two propositions will be found useful in considering some cases of induced electricity.

DEF. A SYSTEM OF EQUIPOTENTIAL SURFACES *denotes a system of surfaces such that the difference of potential between any two consecutive surfaces of the system is constant.*

107. Prop. XVI. In a system of equipotential surfaces the distance of consecutive surfaces increases as the potential diminishes.

Let d be the distance of two consecutive surfaces measured along a line of force, and F the *average* value of the force in direction of a line of force. Then between each successive pair of surfaces Fd is constant, and it is clear that as F diminishes d increases; or the distance of consecutive surfaces grows greater as the force grows less. Now it is clear that force and potential increase or diminish together as we draw nearer to or recede further from attracting matter, and the distance of consecutive surfaces consequently increases as the potential diminishes.

COR. There will be an induced current in any conductor moving near an electrified system. For draw any system of equipotential surfaces, and let a conductor move from the position AB to $A'B'$.

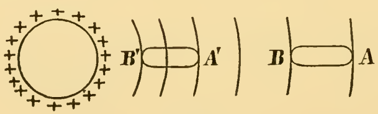

Fig. 29.

It is clear that the difference of potential between A and B is one unit, while that between A', B' is two units. Hence

as the body moves into regions of greater potential, the difference of potential of its ends constantly increases, and to equalize this increasing inequality, a flow of electricity follows in the direction for + electricity from B to A, as long as the movement across equipotential surfaces lasts.

108. Prop. XVII. To calculate the rate of motion of any point on an equipotential surface of given value as the electrification of the system proceeds.

Let AB be an equipotential surface of value V when the charge of the system is M, and let $A'B'$ be the equipotential surface of same value when the system has received a small increase q.

If m be an element of the electrification distant r from Q, the potential at the point Q

$$=\Sigma \frac{m}{r} = V.$$

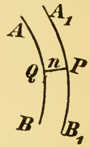

Fig. 30.

If now the charge M receive a small increment q, the density at each point alters in ratio $\frac{q}{M}$.

Hence the potential at Q rises by $\frac{q}{M}V$.

But the potential at P is now V.

Hence the work done on a + unit carried from P to Q

$$=\frac{q}{M}V,$$

and this also equals Fn, where F is the resultant force, and n the length PQ;

$$\therefore Fn = \frac{q}{M}V,$$

$$\text{or } n = \frac{V}{MF}q.$$

Hence the rate of motion of the surface at P varies jointly as the potential and the rate of electrification, and varies inversely jointly as the whole charge and the force at the point.

COR. 1. Since V is represented by $\Sigma \frac{q}{r}$, and F by $\Sigma \frac{q}{r^2} \cos\phi$, where ϕ is the angle between the normal to the surface and r, it is clear that $\frac{V}{F}$ or

$$\frac{\Sigma \frac{q}{r}}{\Sigma \frac{q}{r^2} \cos\phi}$$

will on the whole increase as r on the whole increases, i.e. as V diminishes. Hence, on the whole, the lower equipotential surfaces move during electrification more rapidly than the surfaces at which the potential has a higher value.

COR. 2. It follows from the last Cor. that there will be an induced current in a conductor during the electrification of any neighbouring conductor. For in Fig. 29 the potential surfaces at A are moving more rapidly than those at B. Hence the difference of potential at A and B is constantly increasing, or B's potential relatively to A constantly rising, which determines a flow of electricity from B to A.

109. In the next few Articles we give the investigation of two cases of electrification, the first that of a thin circular plate, the second that of a very thin and long cylinder. We may regard both these cases as the limiting form of a spheroid, which in the first case becomes extremely oblate, and in the second case extremely prolate. We require therefore the following preliminary proposition.

Prop. XVIII. To show that the attraction of a homogeneous shell bounded by two similar and similarly situated spheroids on an internal point vanishes.

Let O be the internal point, and let any cone of small vertical angle cut off from the bounding surfaces the elements Pm and Nq.

Since any section of the shell consists of two similar ellipses, the same diameter will bisect PQ and pq; $\therefore Pp = Qq$, and similarly $Mm = Nn$.

Hence the two small frusta Pm Qn are of the same thickness, and since the vertical angle is small, their masses are proportional to OP^2 and OQ^2, and their attractions on O are inversely in this ratio, and are therefore equal and in opposite directions. The same is true for each pair of small elements similarly described, and the whole shell consequently exerts no attraction on O.

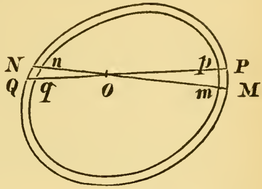

Fig. 31.

This proof is derived from Todhunter's *Analytical Statics*.

110. Prop. XIX. To find the law of density on a freely electrified spheroid.

By the last proposition the law of density is the same as the law of thickness of a very thin material shell bounded by two similar and similarly situated spheroids.

Let AA' be the major axis of generating ellipse, and PP_1 be the thickness of the shell at the point P, then PP_1 produced is the normal to the ellipse at P. Draw PN, P_1QM perpendicular to AA'. Join PQ, and produce it to T, then PQT is the tangent to the ellipse at P, and P_1, Q are corresponding points on the two similar ellipses.

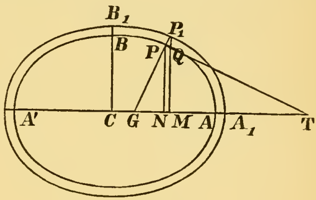

Fig. 32.

Then we have

$$PP_1 = P_1Q \cos PP_1Q = P_1Q \cos GPN = P_1Q \cdot \frac{PN}{PG}.$$

Again $$P_1M^2 = \frac{b'^2}{a'^2}(a'^2 - CM^2),$$

where a', b' = the semi-major and minor axes of the outer ellipse, and $$QM^2 = \frac{b^2}{a^2}(a^2 - CM^2),$$

if a, b denote the semi-axes of the inner ellipse.
On subtracting, since

$$\frac{b^2}{a^2} = \frac{b'^2}{a'^2}, \qquad P_1M^2 - QM^2 = \frac{b^2}{a^2}(a'^2 - a^2) = b'^2 - b^2;$$

$$\therefore\ P_1Q\,(P_1M + QM) = (b' + b)\,(b' - b);$$
$$\therefore\ PN\,.\,P_1Q = b\,(b' - b),$$

remembering that the shell is indefinitely thin.

Hence thickness at $P = \dfrac{b\,(b' - b)}{PG}$.

If we substitute for the material shell an electrical distribution and if ρ_0 represent the density at B the extremity of minor axis,

$$\text{density at } P = \frac{b\rho_0}{PG} = \frac{a\rho_0}{\sqrt{a^2 - e^2x^2}},$$

where $x = CM$, and $e =$ eccentricity of ellipse.

To find the whole quantity of the electrification, the quantity on an element formed by revolution of PQ round the minor axis

$$= 2\pi CN\,.\,PQ\,.\,\frac{b\rho_0}{PG} = 2\pi b\rho_0\,.\,\frac{CN\,.\,PQ}{PG}.$$

But $CG = e^2CN$; $\therefore\ CN = \dfrac{NG}{1 - e^2}$;

$$\therefore\ \text{quantity on elementary annulus} = \frac{2\pi b\rho_0}{1 - e^2}\,.\,\frac{NG\,.\,PQ}{PG}$$
$$= \frac{2\pi b\rho_0}{1 - e^2}\,.\,PQ \sin GPN$$
$$= \frac{2\pi b\rho_0}{1 - e^2}\,(PN - QM),$$

adding the successive values of this difference,

$$\text{the whole amount of electricity} = \frac{2\pi b\rho_0\,.\,2b}{1 - e^2} = 4\pi a^2\rho_0.$$

111. Prop. XX. To find the capacity of a freely electrified thin circular plate.

To deduce the electrification of a circular plate, we notice that a thin circular plate is the limiting form of an oblate spheroid whose minor axis is indefinitely short, while the major axis is finite and equal to a.

In virtue of the relation $b^2 = a^2(1 - e^2)$, we see that in this case we must make $e = 1$.

Hence for the density at any point in terms of the density ρ_0 at the centre, we have

$$\frac{a\rho_0}{\sqrt{a^2 - x^2}} = \rho,$$

where a = radius, and x = distance of point from the centre.

Also the whole quantity is $4\pi a^2 \rho_0$, which shows that the average density is twice the density at the centre.

112. To calculate the potential of a thin circular plate freely electrified, we have only to calculate the potential at the centre of a distribution whose law is given above.

Conceive the plate cut up into a very large number of

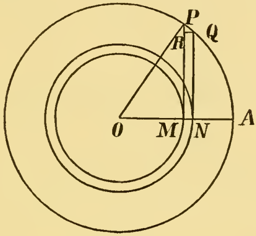

Fig. 33.

narrow annuli, and let OA cut one in M, N. Draw MP, NQ perpendicular to OA, and QR perpendicular to PM, and join OP.

Then the potential at O of the ring MN of the electrification

$$= \frac{2\pi OM . MN}{OM} \cdot \frac{a\rho_0}{\sqrt{OP^2 - OM^2}} = 2\pi a\rho_0 \cdot \frac{MN}{PN}$$

$$= 2\pi a\rho_0 \cdot \frac{PQ}{a} = 2\pi a\rho_0 \,.\, \angle POQ,$$

since the triangles PRQ, OPM are similar and $\frac{PQ}{a}$ is the circular measure of the angle POQ.

Hence adding all the small angles POQ corresponding to successive annuli whose widths lie on OA, we have for the surface :

$$\text{Potential} = 2\pi a\rho_0 \cdot \frac{\pi}{2} = \pi^2 a\rho_0.$$

But we have only yet considered one surface of the plate which will be electrified on both surfaces.

Hence Potential of Plate $V = 2\pi^2 a\rho_0$, and $Q = 4\pi a^2 \rho_0$,

$$\therefore\ V = \frac{\pi}{2a} . Q,$$

when $V = 1$, $Q = C$,

$$\therefore\ C = \frac{2a}{\pi}.$$

Hence

capacity of plate : capacity of sphere of same radius :: 2 : π.

If however the plate forms one conductor of a condenser, the capacity will only be $\frac{a}{\pi}$, since the free charge exists on one side only (see Art. 105).

113. Prop. XXI. To find the capacity of a very long and thin cylinder.

It is clear that, neglecting the ends, the electrification of the cylinder will be sensibly of uniform density. We might see this by viewing the cylinder as the limit of a very prolate spheroid, whose major axis is very large compared to its minor axis. We shall assume the potential everywhere within the cylinder the same as at the middle point of its axis.

Let O be the middle of the axis, and a the radius of the cylinder. Taking an element PQ round the cylinder, the quantity of electricity on its surface is

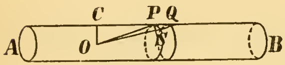

Fig. 34.

$$2\pi a PQ . \rho,$$

and the potential at O of this element

$$= \frac{2\pi a\rho PQ}{OQ}.$$

Draw OC perpendicular to the surface of the cylinder, and PS perpendicular to OQ. Then QSP and QCO are similar triangles.

Hence $\qquad QP : QS :: QO : CQ;$

$$\therefore\ OQ . QS = CQ . PQ;$$

$$\therefore\ \frac{PQ}{OQ} = \frac{QS}{CQ} = \frac{PQ + QS}{OQ + CQ}.$$

Now since when x is a very small fraction,

$$\log(1-x) = -x,$$

we shall commit no appreciable error in putting

$$\begin{aligned}\frac{PQ}{OQ} &= -\log\left(1 - \frac{PQ+QS}{OQ+CQ}\right)\\ &= -\log\frac{OP+CP}{OQ+CQ}\\ &= \log(OQ+CQ) - \log(OP+CP),\end{aligned}$$

which is a form adapted to our method of summation.

The potential of the annulus PQ of the electrification will now

$$= 2\pi a\rho\,\{\log(OQ+CQ) - \log(OP+CP)\},$$

adding all the elementary differences we have for the potential at O of the charge on the cylinder,

$$4\pi a\rho\,\{\log(OB+BC) - \log OC\},$$

and if l be the length of the cylinder, and a its radius, l being assumed great compared to a, this reduces to

$$4\pi a\rho \log\frac{l}{a} = V \text{ suppose,}$$

and if Q be the quantity of electricity, $Q = 2\pi a\rho\,.\,l$;

$$\therefore V = Q\,.\,\frac{2\log\frac{l}{a}}{l}.$$

If $V = 1$, $Q = C$, the capacity;

$$\therefore C = \frac{l}{2\log\frac{l}{a}}.$$

Cor. 1. If l be very large compared to a, $\log\frac{l}{a}$ must be very large, and therefore the capacity of the wire is small. We assume therefore that if two pieces of apparatus be connected by a very fine wire, we may neglect the electrification

of the wire, and that the charge is simply divided in the ratio of the capacities of the apparatus.

Cor. 2. If there be another cylinder outside the one we are considering, separated from it by air, we have for the potential of the Leyden jar so formed,

$$V = \frac{2Q}{l}\left(\log\frac{l}{a} - \log\frac{l}{a'}\right) = \frac{2Q}{l}\log\frac{a'}{a};$$

$$\therefore C = \frac{l}{2\log\dfrac{a'}{a}},$$

a' being the radius of the outer cylinder.

If the dielectric be different from air and have a specific inductive capacity K (Art. 77),

$$C = K\frac{l}{2\log\dfrac{a'}{a}},$$

a formula useful in calculating the charge of a marine cable.

114. We next proceed to illustrate by example the method of electrical images explained in the last chapter (Art. 73).

Prop. XXII. To find the distribution of electricity on an infinite conducting plate connected with the earth and under the influence of an electrified point.

Let A be the electrified point containing m units of electricity, and DE the conducting plate. Draw AF perpendicular to DE, and produce it to B, so that $FB = FA = p$. If we imagine a quantity $-m$ of electricity at B, it is clear that for the system m at A and $-m$ at B, DE will be a surface at zero-potential. For the potential at any point D is $\dfrac{m}{DA} - \dfrac{m}{DB} = 0$; because $DB = DA$

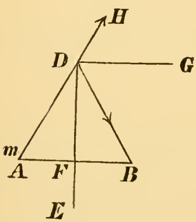

Fig. 35.

Hence we may substitute for the electrification of the plate the charged point B as far as places to the left of DE are concerned.

To find the density at any point on the plate, we have (Art. 72) only to find the resultant force due to the electrified points A and B, and divide by 4π.

This resultant will be normal to the plate, and consists of two components, one repulsive along AD, and one attractive along DB. Hence,

$$\text{Resultant force} = \frac{m}{BD^2}\cos BDG + \frac{m}{AD^2}\cos HDG$$

$$= \frac{2m}{AD^2}\cos DAF = \frac{2m}{AD^2}\cdot\frac{AF}{AD} = \frac{2mp}{AD^3},$$

and is directed to the right of DE. Hence the density will be negative, and at point D

$$= -\frac{2mp}{4\pi AD^3} = -\frac{mp}{2\pi AD^3},$$

or the density varies inversely as the cube of the distance from the electrified point.

This method can be applied to the electrification, under the influence of an electrified particle, of any system of planes, by means of the optical principle of successive images, and the electrification can be represented in finite terms whenever the number of such images is finite: we give a few cases as examples:—

115. *Example* 1. Two planes at right angles to each other. Let OA, OB be conducting planes at right angles to

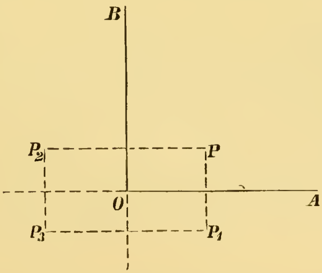

Fig. 36.

each other, P a point electrified with one unit of + electricity. The successive images of P are P_1 in OA, P_2 in OB, and P_3 the image of both P_2 in OA and P_1 in OB. Let there be quantities of electricity -1 at P_1 and P_2 and $+1$ at P_3. Then the potential at any point on either plane due to P and its system of images vanishes, and the potential at any point within the angle BOA will be that due to P combined with the system of images P_1, P_2, P_3.

The density of the electrification at any point Q in OA will be that due to the superposition of the systems (P, P_1) and (P_2, P_3). Therefore by preceding article,

$$\rho = -\frac{p}{2\pi}\left(\frac{1}{QP^3} - \frac{1}{QP_2^3}\right),$$

p being the distance of P from OA.

116. *Example* 2. Three planes mutually at right angles. Let OA, OB be two of the intersections of these planes and OC, perpendicular to the plane of the paper, the third intersection; P not being now in the plane of the paper. There will in this case be the system of images P_1, P_2, P_3 due to the planes COA, COB and the images of P, P_1, P_2, P_3 in the plane AOB. The density at any point Q in the plane AOB will be given by

$$\rho = -\frac{p}{2\pi}\left(\frac{1}{QP^3} - \frac{1}{QP_1^3} + \frac{1}{QP_3^3} - \frac{1}{QP_2^3}\right),$$

p being the distance of P from the plane AOB.

117. *Example* 3. Two parallel planes with an electrified point between them. In this case the number of images is infinite, and it will not be generally possible to represent the density in finite terms. In any practical case the influence of all images after the first few could be neglected. The positions of the images are easily found. Let A, B be the plates, and P the point between them distant a, b from them respectively, at which we will suppose a unit of electricity placed.

Let P_1 be the image of P in A, P_2 that of P_1 in B, P_3 that of P_2 in A and so on. The set of images P_1, P_3...

to the right of A will be negative, and the set P_2, P_4... to the left of B will be positive, the quantity in all cases being unity.

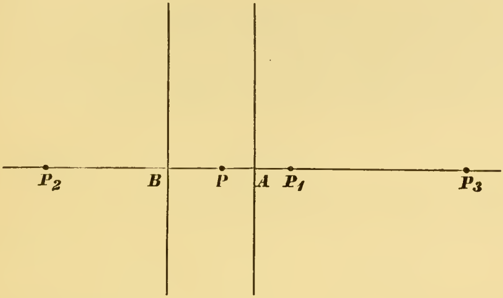

Fig. 37.

We have

$$PP_1 = 2a, PP_2 = PP_1 + 2b, PP_3 = PP_2 + 2a, PP_4 = PP_3 + 2b, \&c.$$

$$\therefore \quad PP_1 = 2a, \quad PP_2 = 2a + 2b, \quad PP_3 = 4a + 2b, \quad PP_4 = 4a + 4b.$$

Hence generally

$$PP_{2x} = 2x(a+b), \quad PP_{2x-1} = 2a + 2(x-1)(a+b),$$

where x may have any value between 1 and ∞.

If we start with the image of P in B, which we will call Q_1, we shall have the set of images Q_1, Q_3... to the left of B and negative, and the set Q_2, Q_4... to the right of A and positive. Their positions being given by

$$PQ_{2x} = 2x(a+b), \quad PQ_{2x-1} = 2b + 2(x-1)(a+b).$$

Hence on the whole we have to the right of A,

The set of $-$ images at distances from P $\quad 2a + 2(x-1)(a+b)$

and the set of $+$ images $\quad$ „ $\quad 2x(a+b)$

To the left of B,

The set of $-$ images at distances from P $\quad 2b + 2(x-1)(a+b)$

and the set of $+$ images $\quad$ „ $\quad 2x(a+b)$

x in all cases being any integer from 1 to ∞, and the quantity of electricity in each image being ± 1.

118. **Prop. XXIII. To find the distribution of electricity on a sphere at zero potential under the influence of an electrified point within or without it.**

We may deduce the electrical influence of a point on a sphere by considering two points having charges e_1 and $-e_2$ of electricity placed at points A, B. We shall have for the potential at any point distant r_1 and r_2 from A, B respectively

$$\frac{e_1}{r_1} - \frac{e_2}{r_2}.$$

Hence for surface of zero-potential

$$\frac{e_1}{r_1} = \frac{e_2}{r_2}, \text{ or } r_1 : r_2 :: e_1 : e_2.$$

Or, the distances of any point on the surface from A and B are in the constant ratio e_1 to e_2.

We can easily show that the locus of a point satisfying this condition is a sphere.

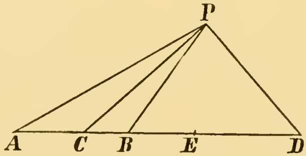

Fig. 38.

For let P be a point on the locus, and e_1 greater numerically than e_2. Divide AB internally and externally in C and D, so that

$$AC : CB :: AD : DB :: e_1 : e_2 \text{............(i).}$$

Hence $AP : PB :: AC : CP$; $\therefore$ PC bisects APB,

and $AP : PB :: AD : DB$; $\therefore$ PD bisects APB externally;

$\therefore$ CPD is a right angle.

Hence the locus of P on the plane of the paper will be a circle whose diameter is CD: and the same property will be true for each point on the sphere whose diameter is CD and centre E, the middle point of CD.

119. To determine the position and dimensions of the sphere we have the following relations.

From (i) we have

$$AC + CB : CB :: e_1 + e_2 : e_2;$$

$$\therefore\ CB = \frac{e_2}{e_1 + e_2} AB, \text{ and } AC = \frac{e_1}{e_1 + e_2} AB \text{........ (ii).}$$

Again from (i),

$$AD - DB : DB :: e_1 - e_2 : e_2,$$

$$\therefore\ DB = \frac{e_2}{e_1 - e_2} AB, \text{ and } AD = \frac{e_1}{e_1 - e_2} AB \text{......(iii).}$$

Also $CD = CB + BD = \dfrac{e_2}{e_1 + e_2} AB + \dfrac{e_2}{e_1 - e_2} AB = \dfrac{2e_1e_2}{e_1^2 - e_2^2} . AB.$

$$\text{Hence radius of circle} = \frac{e_1e_2}{e_1^2 - e_2^2} . AB \text{..................(iv).}$$

And $EB = CE - CB = \dfrac{e_1e_2}{e_1^2 - e_2^2} . AB - \dfrac{e_2}{e_1 + e_2} AB = \dfrac{e_2^2}{e_1^2 - e_2^2} AB$

$$= \frac{e_2}{e_1} EC.$$

Similarly $$EA = \frac{e_1}{e_2} EC;$$

$$\therefore\ EB . EA = EC^2.$$

A and B are called conjugate points with reference to the sphere.

120. To find the density at any point, we have (Art. 72), to find the resultant force at that point on the spherical surface, and as before divide it by 4π.

The forces at P are clearly $\dfrac{e_1}{r_1^2}$ along AP, and $\dfrac{e_2}{r_2^2}$ along PB, and their resultant is along PE, since it is normal to

the equipotential surface. Let F be the magnitude of the resultant, then completing the parallelogram $EKPL$,

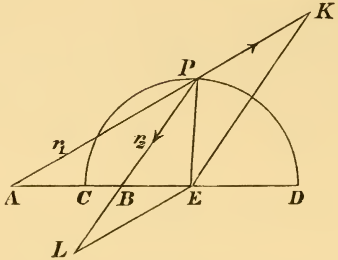

Fig. 39.

$$F : \text{component in } AP :: PE : PK,$$

$$\text{or } F = \frac{PE}{PK}\frac{e_1}{r_1^2},$$

and because PB, KE are parallel,

$$PK : BE :: AP : AB,$$

$$\therefore\ PK = \frac{BE}{AB}.AP,$$

substituting; $F = \dfrac{AB.PE}{BE}.\dfrac{e_1}{r_1^3} = a.\dfrac{e_1^2 - e_2^2}{e_2^2}.\dfrac{e_1}{r_1^3}$,

if a be the radius of the sphere.

Since this resultant is inwards, we must express the density by

$$-\frac{e_1(e_1^2 - e_2^2)}{4\pi e_2^2}.\frac{a}{r_1^3},$$

and we conclude that a distribution whose density is given by the above law produces within the sphere a force equal and opposite to e_1 at A, and is therefore the distribution induced by e_1 placed at A.

Again, if the density at P be expressed by

$$+\frac{e_2(e_1^2 - e_2^2)}{4\pi e_1^2}\frac{a}{r_2^3},$$

we conclude that this distribution produces without the sphere a force equal and opposite to that of $-e_2$ at B, and is therefore the distribution induced by $-e_2$ at B.

The whole quantity of the distribution being in either case $\pm e_2$.

(i) In the *first case* let $AE=f$, then $BE=\frac{a^2}{f}$,

$$\text{and } f=\frac{e_1}{e_2}a.$$

Hence the law of density becomes, on substitution for e_2 and reduction,

$$-\frac{e_1(f^2-a^2)}{4\pi a\,.\,r_1^3}.$$

(ii) In the *second case* let $BE=f'$, then $AE=\frac{a^2}{f'}$,

$$\text{and } f'=\frac{e_2}{e_1}a.$$

Hence substituting for e_1 the law of density is

$$+\frac{e_2(a^2-f'^2)}{4\pi a r_2^3}.$$

121. We see now that we can include both cases in the following statement:

If there be taken on the radius of a sphere two conjugate points, and a quantity of electricity e be placed at one of them whose distance from the centre is f, it will induce a distribution over the sphere whose law of density is

$$-\frac{(a^2\sim f^2)\,e}{4\pi a r^3},$$

where $r=$ distance from the electrified point

and $a=$ radius of sphere,

and the resultant effect of this distribution on all points on the same side of the spherical surface is the same as of a particle at the conjugate point having a charge represented

by $-\dfrac{a}{f}e$; the whole quantity of the distribution being, when the electrified point is within the sphere, $-e$, and when without the sphere, $-\dfrac{a}{f}e$.

122. The following direct geometrical proof of the proposition of the preceding article is a modification of that originally given by Sir William Thomson.

Prop. XXIV. A distribution of matter is made over a spherical surface whose density at any point varies inversely as the cube of its distance from a fixed point, show that the potential of the distribution at any point on the opposite side of the spherical surface is the same as that due to a certain quantity of matter at the given fixed point.

Suppose S the given fixed point in Fig. (a) external, and in Fig. (b) internal, and let P be a point, on the opposite side of the spherical surface, at which we shall estimate the potential.

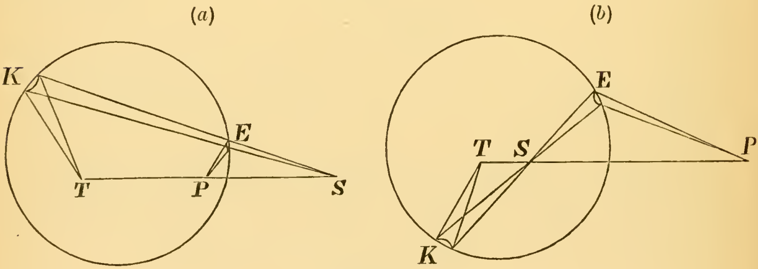

Fig. 40.

Let in Fig. (a) the distance of S from the centre be f, and in Fig. (b) f'.

Join SP and produce it to T so that

$$\text{Fig. } (a) \quad SP \, . \, ST = f^2 - a^2,$$

$$\text{Fig. } (b) \quad SP \, . \, ST = a^2 - f'^2.$$

Through S draw any line meeting the sphere in E and K, join PE and TK;

$$\therefore\ SE\,.\,SK = SP\,.\,ST \text{ in both figures;}$$

$$\therefore \text{ the triangles } SEP,\ STK \text{ are similar.}$$

Conceive now the line KES to move so as to trace out a small cone whose vertex is S, and which cuts the spherical surface in elementary areas E and K. It is clear that E and K are corresponding elements, so that the whole surface is exhausted *simultaneously* by a series of elements E and K.

Now the potential at the point P due to the element E of the distribution (E being used to denote its area),

$$= \frac{\text{mass of } E}{EP},$$

and density over E $\qquad = \dfrac{\lambda}{SE^3}$, where λ is a constant;

$$\therefore \text{ potential at } P \text{ due to } E = \frac{\lambda E}{EP\,.\,SE^3}.$$

Again, since the tangent planes at E, K are equally inclined to SEK,

$$E : K :: SE^2 : SK^2;$$

$$\therefore \text{ potential due to } E = \frac{\lambda K}{EP\,.\,SE\,.\,SK^2} = \frac{\lambda K}{EP\,.\,SK\,(f^2 \sim a^2)};$$

also by similar triangles $EP : SP :: TK : SK$,

$$\therefore\ EP\,.\,SK = SP\,.\,TK,$$

$$\therefore \text{ potential due to } E = \frac{\lambda K}{SP\,(f^2 - a^2)\,.\,TK};$$

$$\therefore \text{ potential at } P \text{ due to whole sphere} = \frac{1}{SP\,(f^2 - a^2)}\,.\,\Sigma\,\frac{\lambda K}{TK}.$$

But $\Sigma\,\dfrac{\lambda K}{TK}$ represents the potential at T of a uniform distribution whose density is λ, which by Prop. 1 is $4\pi\lambda a$, since T is necessarily an internal point.

8—2

Hence the potential at P, due to the sphere $= \dfrac{4\pi\lambda a}{(f^2 \sim a^2)\, SP}$

$= $ potential at P, due to a mass $\dfrac{4\pi\lambda a}{f^2 \sim a^2}$ at S,

or substituting electric distribution for matter, and reversing the sign of the distribution on the sphere,

Potential due to quantity $\dfrac{4\pi\lambda a}{f^2 \sim a^2}$ at $S +$ Potential due to the distribution of density $-\dfrac{\lambda}{SE^3} = 0$, at any point P on the opposite side of the surface to S.

And this is the condition which must be satisfied by the distribution induced by an electrified particle at S (Art. 72).

If we put $$m = \frac{4\pi\lambda a}{f^2 \sim a^2}, \text{ we get}$$

Potential due to m placed at $S +$ Potential due to distribution of density $-\dfrac{(f^2 \sim a^2)\, m}{4\pi a\,.\, SE^3} = 0$, at any point on the opposite side of the surface.

To find the whole quantity distributed over the sphere we see

$$\text{quantity on element } E = \frac{\lambda E}{SE^3} = \frac{\lambda K}{SE\,.\,SK^2} = \frac{\lambda K}{(f^2 \sim a^2)\, SK};$$

$$\therefore \text{ quantity distributed } = \frac{1}{f^2 \sim a^2} \Sigma \frac{\lambda K}{SK},$$

and $\Sigma \dfrac{\lambda K}{SK} =$ potential at S of a uniform distribution of density λ.

In Figure (a) S is external,

$$\therefore \Sigma \frac{\lambda K}{SK} = \lambda \frac{4\pi a^2}{f},$$

and the whole amount of distribution $= \dfrac{\lambda 4\pi a^2}{f(f^2 - a^2)}$

$$= \frac{a}{f} m.$$

In Figure (*b*) S is internal, and

$$\therefore \Sigma \frac{\lambda K}{SK} = \lambda \,.\, 4\pi a;$$

$$\therefore \text{ whole amount of distribution } = \frac{\lambda \,.\, 4\pi a}{a^2 - f^2}$$

$$= m,$$

as has been already shown.

By choosing conjugate points, it is easy to show that the two distributions, one derived from the external and the other from the internal point, are identical, and the proposition of Art. 121 follows immediately.

123. Prop. XXV. To find the quantity of the electricity induced on two concentric spherical surfaces each at zero potential, having an electrified point between them.

This can clearly be solved by means of the principle of successive images illustrated above in the case of plane surfaces (see Art. 117).

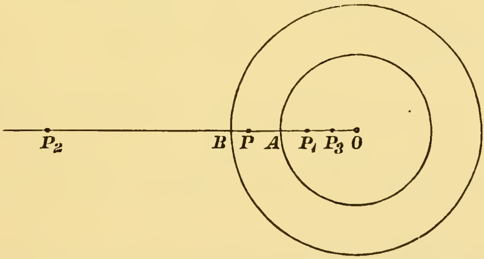

Fig. 41.

Let OAB be a common radius on which P the point charged suppose with a unit of electricity lies.

The positions of the successive images beginning with that in the surface A, which we will call P_1, are given by the formulæ

$$OP_1 \,.\, OP = OA^2 : OP_2 \,.\, OP_1 = OB^2 : OP_3 \,.\, OP_2 = OA^2, \text{ \&c.,}$$

or generally,

$$OP_{2x+1} . OP_{2x} = OA^2 : OP_{2x} . OP_{2x-1} = OB^2.$$

The quantity of electricity in the successive images will be given (Art. 121) by the formula $P_r = -\dfrac{R}{OP_{r-1}} . P_{r-1}$, where R is the radius of the circle with respect to which the image is taken. Hence the quantity

$$\text{at } P_1 = -\frac{OA}{OP},$$

$$\text{at } P_2 = +\frac{OB}{OP_1} \cdot \frac{OA}{OP} = \frac{OB}{OA};\ \because OP_1 . OP = OA^2,$$

$$\text{at } P_3 = -\frac{OA . OB . OA}{(OP_2 . OP_1)\, OP} = -\frac{OA}{OB} \cdot \frac{OA}{OP},$$

$$\&c. \qquad = \qquad \&c.$$

$$\text{at } P_{2x-1} = -\frac{OA . OB \ldots\ldots OA}{(OP_{2x-2} . OP_{2x-3}) \ldots\ldots (OP_2 . OP_1) . OP}$$

$$= -\frac{OA^x . OB^{x-1}}{OB^{2x-2} . OP} = -\left(\frac{OA}{OB}\right)^x \cdot \frac{OB}{OP},$$

$$\text{at } P_{2x} = +\frac{OB . OA . OB \ldots\ldots OA}{(OP_{2x-1} . OP_{2x-2}) \ldots\ldots (OP_1 . OP)} = \left(\frac{OB}{OA}\right)^x,$$

the images of type P_{2x-1} being negative and within A, those of type P_{2x} being positive and outside B.

Beginning with Q_1 the image of P in B, and proceeding as above, we shall have a series of images of types,

$$Q_{2x} = \left(\frac{OA}{OB}\right)^x : Q_{2x-1} = -\left(\frac{OB}{OA}\right)^x \cdot \frac{OA}{OP},$$

those of type Q_{2x} being positive and within A, those of type Q_{2x-1} negative and outside B.

The quantity of electricity induced on A will be found by summing the quantities of all the images within A.

$$\text{This sum} \quad = -\Sigma_1^\infty \left(\frac{OA}{OB}\right)^x \cdot \frac{OB}{OP} \cdot + \Sigma_1^\infty \left(\frac{OA}{OB}\right)^x$$

$$= -\left(\frac{OB}{OP} - 1\right) \Sigma_1^\infty \left(\frac{OA}{OB}\right)^x$$

$$= -\frac{PB}{OP} \cdot \frac{\frac{OA}{OB}}{1 - \frac{OA}{OB}} = -\frac{PB}{OP} \cdot \frac{OA}{AB}.$$

The series for the images outside B is divergent. The sum however must be finite, and by Art. 54,

$$= -\left(1 - \frac{PB}{AB} \cdot \frac{OA}{OP}\right) = -\frac{PA}{OP} \cdot \frac{OB}{AB}.$$

Cor. Supposing the spheres of infinite radius we get the induction on two parallel plates A, B due to an electrified point between them, equal to $-\dfrac{PB}{AB}$ on A and $-\dfrac{PA}{AB}$ on B.

124. A very large and important class of electrical problems are solved by the method of electrical images combined with the principle known to mathematicians as that of geometrical inversion. We must therefore give first the fundamental properties of geometrical inversion.

Def. Centre of Inversion. Geometrical Image or image by inversion. *Let* O *be a fixed point, the centre of inversion, and let* A *be any point in space. Join* OA *and in it take another point* a *such that* OA . Oa = R², *then* a *is said to be the geometrical image of* A, *and* R *is called the radius of the circle of inversion.*

We see that to every curve or surface in space will correspond another curve or surface which will be called its image by inversion, being the locus of the images by inversion of the points on the curve or surface.

125. The following properties of inversion will be found useful.

Prop. A. If O be the centre of inversion, AB any two points, a, b their images, then the triangles OAB, Oba are similar.

For, since $OA \,.\, Oa = OB \,.\, Ob = R^2$,

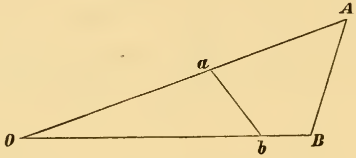

Fig. 42.

$$OA : OB :: Ob : Oa,$$

and the angle at O is common, therefore the triangles are similar.

COR. $AB : ab = OA : Ob = OB : Oa = OA \,.\, OB : R^2$
$$= R^2 : Oa \,.\, Ob.$$

126. Prop. B. The angle at which two curves or surfaces cut is unaltered by inversion.

For let AB, AC represent the elements of two arcs cutting at A, supposing them both in the plane of the paper, the only case that occurs in our work.

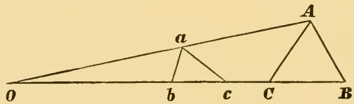

Fig. 43.

Then $\quad \angle OAB = \angle Oba$; and $\angle OAC = \angle Oca$;
$$\therefore \angle BAC = \angle Oba - Oca = \angle bac.$$

127. Prop. C. Corresponding elements of arc in a curve and its image are in the ratio $OA^2 : R^2$ and corresponding elements of surface in the ratio $OA^4 : R^4$.

For, by Prop. A. Cor.,

$$AB : ab :: OA \,.\, OB : R^2.$$

But since A, B nearly coincide $OA \cdot OB = OA^2$ nearly;

$$\therefore AB : ab :: OA^2 : R^2.$$

Again, elements of surface are ultimately proportional to rectangles under elements of arc, and hence will be in the duplicate ratio of arcs or as $OA^4 : R^4$, or as $R^4 : Oa^4$.

128. Prop. D. The image of a plane is a sphere passing through the centre of inversion.

Let OA be the perpendicular on the plane and P any point on it. Then if a, p be the images of A, P,

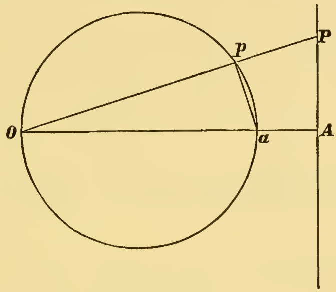

Fig. 44.

$$\angle Opa = \angle OAP = \text{a right angle},$$

$\therefore$ locus of p is a sphere on Oa as diameter.

129. Prop. E. The image of a sphere is another sphere, the two spheres having the point of inversion for centre of similitude.

Let $OECE'$ be drawn through the centre C of the sphere,

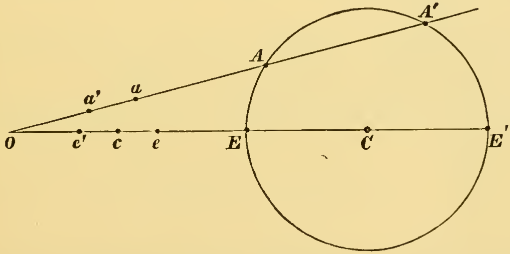

Fig. 45.

and let OAA' be any other line cutting the sphere. Then using in all cases small letters for the images of corresponding capital letters,

$$OA \,.\, Oa = R^2, \text{ and } OA \,.\, OA' = OC^2 - CE^2 = f^2 - a^2, \text{ suppose};$$

$$\therefore \frac{Oa}{OA'} = \frac{R^2}{f^2 - a^2}, \text{ and similarly } \frac{Oa'}{OA} = \frac{R^2}{f^2 - a^2}.$$

Hence the locus of a' is similar to that of A, or the image is another sphere, the convex part of the image corresponding to the concave part of the sphere, and *vice versâ.*

If $Oc = f'$ and $ce = a'$, we shall have

$$\frac{f'}{f} = \frac{a'}{a} = \frac{R^2}{f^2 - a^2}.$$

130. To apply this geometry to electrical problems, we must define the electrical image of an electric distribution.

Def. Electrical Image by Inversion. *Let there be at a point* B *a quantity* e *of electricity, then the quantity* e′ *placed at* b, *the geometrical image of* B, *will be called the electrical image by inversion of* e *if* $\frac{e'}{e} = \frac{R}{OB} = \frac{Ob}{R}$, R *being the radius of the sphere of inversion.*

This of course is only an extension of the electrical images discussed above, Art. 121. We can now prove the following important proposition.

131. Prop. XXVI. Relation between the potential of any electrical distribution and that of its image.

Let B (Fig. 42) be an element of an electric distribution whose quantity is e; b its electrical image at which is a quantity $e'\left(= \frac{Ob}{R}e\right)$, and let A be any point, and a its image.

Then if V, V' be respectively the potential, due to B, at A, and the potential, due to b, at a, we have

$$V : V' = \frac{e}{AB} : \frac{e'}{ab} \text{ or } \frac{V'}{V} = \frac{e'}{e} \cdot \frac{AB}{ab} = \frac{Ob}{R} \cdot \frac{R^2}{Oa \, . \, Ob} \text{ (Prop. A)};$$

$$\therefore \; V' = \frac{RV}{Oa} = \text{potential at } a \text{ of a quantity } RV \text{ at } O.$$

The same proposition will be true for each element of any distribution. Hence by the principle of superposition if V be the potential at A of any electrical system B, then the potential of the image of B at the image of A will equal the potential of a quantity RV placed at the centre of inversion.

COR. 1. If we place at O the centre of inversion a quantity of electricity, $-RV$, then the potential at a of the system b, and of a quantity $-RV$ at O, will be zero. If B represent therefore any electrified conductor, and A a point inside it or on its surface, we obtain from it at once the distribution induced by an electrified point O on the image of B kept at zero potential.

COR. 2. If we have given the distribution of electricity on any conductor B at zero potential under the influence of an electrified particle at O: by inverting the system relatively to O, the image of O is at an infinite distance, and can therefore be neglected, while the added charge at O will be zero because B is at zero potential. We have left therefore the distribution on the image of B freely electrified.

COR. 3. If σ, σ' be corresponding elements of area of B, and its image b, we have (Prop. C) $\frac{\sigma}{\sigma'} = \frac{R^4}{Ob^4}$, and if e, e' be quantities of electricity at B and b, $\frac{e}{e'} = \frac{R}{ob}$, hence if ρ, ρ' be densities at corresponding points on an electrical distribution and its image we have

$$\frac{\rho'}{\rho} = \frac{e'}{e} \times \frac{\sigma}{\sigma'} = \frac{R^3}{Ob^3}.$$

We proceed to apply the principle of inversion to the solution of certain problems.

132. Prop. XXVII. To find the electrification of a sphere at zero potential under the influence of an electrified particle.

1st. Let the particle be external.

Take a sphere freely electrified in space to potential V, and invert it, making any external point centre of inversion. We will use the notation and figure of Prop. E. The original electrical density was $\frac{V}{4\pi a}$ at A suppose:

$$\therefore \text{ density on the image} = \frac{R^3}{Oa^3} \times \frac{V}{4\pi a} = \frac{VR^3}{4\pi a Oa^3}.$$

If e be the quantity of electricity at centre of inversion

$$e = -VR;$$

$$\therefore \text{ density on image} = \frac{-eR^2}{4\pi a Oa^3}.$$

But
$$\frac{a'}{a} = \frac{R^2}{f^2 - a^2};$$

$$\therefore \frac{R^2}{a} = a' . \frac{f^2 - a^2}{a^2} = a' . \frac{f'^2 - a'^2}{a'^2} = \frac{f'^2 - a'^2}{a'};$$

$\therefore$ electrical density $= -\frac{e(f'^2 - a'^2)}{4\pi a' . r^3}$, if $r(= Oa)$ be the distance of the electrified point.

2nd. Let the influencing point be internal.

To arrive at this, we will invert the distribution of the preceding case, making the electrified sphere the sphere of inversion. The distribution on the sphere is unchanged, since the sphere is its own image; the image of the electrified point is a point distant $\frac{a'^2}{f'}$ $(=g')$ from the centre, and its charge is $\frac{g'}{a'}e\,(=e')$. Hence the system consisting of e' at distance g' from centre, and the distribution according to density law $-\frac{e(f'^2 - a'^2)}{4\pi a' r^3}$, will give zero potential at all

external points. The total quantity of distribution

$$= e' = \frac{g'}{a'} e = \frac{a'}{f'} e.$$

Substituting for f' and e their values in terms of e', g' we find the density induced by e' at a distance g' from the centre $= -\frac{e'(a'^2 - g'^2)}{4\pi a' \, . \, r'^3}$, where r' is the distance from the influencing particle of the element of the distribution, remembering (Art. 118) that

$$\frac{r'}{r} = \frac{a'}{f'} = \frac{g'}{a'}.$$

133. Prop. XXVIII. To find the electrical distribution on a conductor in the form of two spherical surfaces which cut at right angles.

We shall obtain this system by the inversion of the system (Art. 115) of two conducting planes at right angles to each other, since each plane inverts into a sphere and the angle between the spheres is the same as between the planes.

Let P, the electrified particle, be the centre of inversion,

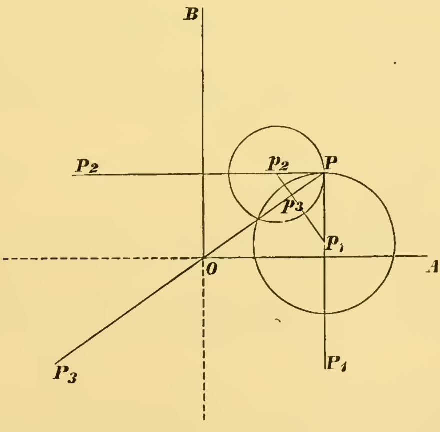

Fig. 46.

and the radius of the sphere of inversion unity. The radii of spheres which are the images of OA, OB will be

$$\frac{1}{2a}(=\alpha) \text{ and } \frac{1}{2b}(=\beta),$$

a, b being the distances of P from OA, OB.

The distances of p_1, p_2, p_3 the images of P_1, P_2, P_3 from P will be

$$Pp_1=\frac{1}{2a}(=\alpha),\ Pp_2=\frac{1}{2b}(=\beta) \text{ and } Pp_3=\frac{1}{2\sqrt{a^2+b^2}}\left(=\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}\right),$$

and the quantities of electricity will (Art. 130) be equal to their distances from P (R by supposition being unity); the signs being $-$ for p_2 and p_1, and $+$ for p_3; and this system of images will be equivalent to the electrified conductor at all external points. Also the potential at P due to each image will be numerically unity, and therefore the potential at P due to the whole system of images will be -1.

The quantity of electricity on the conductor will equal the sum of the quantities in the images p_1, p_2, p_3

$$=-\alpha-\beta+\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Hence the capacity of the conductor or the quantity which will keep it at unit potential

$$=\alpha+\beta-\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}.$$

134. Prop. XXIX. To find the electrification of a conductor in the form of three spheres cutting mutually at right angles.

This will be obtained by the inversion of the system (Art. 116) of three conducting planes, mutually at right angles.

The distances from P of the seven images in the planes AB, BC, CA are easily seen to be (if a, b, c be the distances of P from the planes)

(a) $2a$, $2b$, $2c$ at which are quantities -1,

(b) $2\sqrt{a^2+b^2}$, $2\sqrt{b^2+c^2}$, $2\sqrt{c^2+a^2}$ at which are quantities $+1$,

(c) $2\sqrt{a^2+b^2+c^2}$ at which is a quantity -1.

The radii of the spheres formed by inversion of the system of planes in a sphere of radius unity

$$=\frac{1}{2a},\ \frac{1}{2b},\ \frac{1}{2c}.$$

The distances of the electrical images of the above system (a), (b), (c), will be

(a) $$\frac{1}{2a},\ \frac{1}{2b},\ \frac{1}{2c},$$

(b) $$\frac{1}{2\sqrt{a^2+b^2}},\ \frac{1}{2\sqrt{b^2+c^2}},\ \frac{1}{2\sqrt{c^2+a^2}},$$

(c) $$\frac{1}{2\sqrt{a^2+b^2+c^2}}.$$

At these the quantities of electricity will be numerically the same as their distances, the first row (a) being $-$, the second (b) $+$, and the third (c) $-$.

Hence the quantity of electricity corresponding to unit potential

$$=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}-\frac{1}{2\sqrt{a^2+b^2}}-\frac{1}{2\sqrt{b^2+c^2}}-\frac{1}{2\sqrt{c^2+a^2}}$$

$$+\frac{1}{2\sqrt{a^2+b^2+c^2}}$$

$$=\alpha+\beta+\gamma-\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}-\frac{\beta\gamma}{\sqrt{\beta^2+\gamma^2}}-\frac{\gamma\alpha}{\sqrt{\gamma^2+\alpha^2}}$$

$$+\frac{\alpha\beta\gamma}{\sqrt{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}},$$

if α, β, γ be the radii of the spheres.

135. Prop. XXX. To find the electrification of a system consisting of two spheres touching each other.

This problem will be the electrical inversion of the case (Art. 117) of two parallel planes with an electrified point between them, the centre of inversion being the electrified point.

The images to the right of A (Fig. 37) will have their images within the sphere A', which is the image by inversion of A; while those to the left will fall in the sphere B'; and as before the capacity of each sphere will be given by changing the sign of the sum of the quantities at each electric image.

Thus if C_A be the capacity of A'

$$-C_A = \Sigma_1^\infty \frac{1}{2x(a+b)} - \Sigma_1^\infty \frac{1}{2a + 2(x-1)(a+b)},$$

the symbol Σ_1^∞ denoting that the terms formed by giving x all values from 1 to ∞ are to be added;

$$\therefore\ C_A = \frac{b}{a+b} \Sigma_1^\infty \frac{1}{x\{2a + 2(x-1)(a+b)\}},$$

and similarly

$$C_B = \frac{a}{a+b} \Sigma_1^\infty \frac{1}{x\{2b + 2(x-1)(a+b)\}}:$$

if α, β be the radii of the two spheres $\alpha = \frac{1}{2a}$, $\beta = \frac{1}{2b}$,

$$\therefore\ C_A = \frac{\alpha^2\beta}{\alpha+\beta} \Sigma_1^\infty \frac{1}{x\{x(\alpha+\beta)-\alpha\}},$$

$$C_B = \frac{\alpha\beta^2}{\alpha+\beta} \Sigma_1^\infty \frac{1}{x\{x(\alpha+\beta)-\beta\}}.$$

These summations cannot be generally effected. The two simplest cases are (1) where the spheres are equal,

$$C_A = C_B = \alpha\Sigma_1^\infty \frac{1}{2x(2x-1)} = \alpha\Sigma_1^\infty \left(\frac{1}{2x-1} - \frac{1}{2x}\right)$$

$$= \alpha\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \&c.\right) = \alpha \log_e 2.$$

(2) When β is very small compared with α. C_A appears to be indeterminate, but can be found thus,

$$C_A = \frac{\alpha^2\beta}{\alpha+\beta}\Sigma_1^\infty \frac{1}{x\{x(\alpha+\beta)-\alpha\}} = \frac{\alpha^2\beta}{\alpha+\beta}\left[\frac{1}{\beta}+\Sigma_2^\infty \frac{1}{x\{x(\alpha+\beta)-\alpha\}}\right]$$

$$= \frac{\alpha^2}{\alpha+\beta}+\frac{\alpha\beta}{\alpha+\beta}\Sigma_2^\infty \frac{1}{x(x-1)},$$

neglecting β compared with $(x-1)\alpha$ when $x>1$.

$$\text{And } \Sigma_2^\infty \frac{1}{x(x-1)} = \Sigma_2^\infty\left(\frac{1}{x-1}-\frac{1}{x}\right) = \tfrac{1}{1}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\&c. = 1.$$

Hence to the same degree of approximation $C_A = \alpha$, the same as if the small sphere were not present. Using the same approximation,

$$C_B = \frac{\beta^2}{\alpha}\Sigma_1^\infty \frac{1}{x^2} \text{ very nearly,}$$

$$= \frac{\beta^2}{\alpha}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c.\right) = \frac{\beta^2}{\alpha}\cdot\frac{\pi^2}{6},$$

as is proved in Trigonometry.

To compare the densities on the two spheres divide the charge of each by its area and we have

$$\rho_\alpha = \frac{\alpha}{4\pi\alpha^2} = \frac{1}{4\pi\alpha},$$

$$\rho_\beta = \frac{\pi^2\beta^2}{24\pi\alpha\beta^2} = \frac{\pi}{24\alpha};$$

$$\therefore \frac{\rho_\beta}{\rho_\alpha} = \frac{\pi^2}{6} = 1{\cdot}645 \text{ nearly.}$$

We may assume therefore that when a small sphere is made to touch any electrified conductor it carries away an electrification whose density is equal to 1·645 times the density at the point of contact.

136. By inverting the system (Art. 111) consisting of a freely electrified plate Sir W. Thomson has found the electrifi-

cation of a spherical bowl, due to the induction of an electrified particle anywhere on its surface produced. By inverting the electrification (Art. 123) of two concentric spherical surfaces under the influence of a point between them, he has deduced the electrification of two spherical conductors under each other's induction. For these and other problems, requiring for their solution higher mathematical analysis, we refer the student to the writings of Sir W. Thomson or Clerk Maxwell.

Examples on Chapters III. and IV., and on General Statical Electricity.

1. Two particles are charged with quantities q_1 and q_2 of electricity, and another with a quantity $-(q_1+q_2)$, and are placed at the angular points of a triangle. Show that the work done against the two former equals that done by the latter in bringing a + unit up to the centre of the circumscribing circle.

2. Three particles are charged with equal quantities, two + and one −, of electricity. Show that the centre of the inscribed circle of the triangle, formed by the three particles, will be on the surface of zero potential if

$$4 \sin \frac{\pi - A}{4} \cdot \sin \frac{\pi + B}{4} \cdot \sin \frac{\pi + C}{4} = 1,$$

the negatively electrified particle being at A.

3. A rhombus is constructed, two of whose angles are 60°, and a + unit of electricity is placed at each. Two − units are placed at one of the other angles. Show that the potential at the remaining angle is zero.

4. A funnel drawn out into a capillary tube is filled with sulphuric acid, and a gold leaf electroscope having a gold cap is placed underneath it. A rod of sealing-wax, which has been rubbed with gun-cotton, is now held over the funnel; the acid flows out on to the cap of the electroscope and the leaves diverge. Explain the electrical actions

which produce the flow of liquid and the divergence of the leaves.

5. An insulated metal lamp is placed in a room in which an electrical machine is at work. The lamp is connected by a wire with a gold leaf electroscope in an adjoining room.

(i) Describe the indications of the electroscope after lighting the lamp and working the machine.

(ii) Describe the indications of the electroscope after the lamp is blown out and the machine stopped.

(iii) An insulated metal cylinder completely encloses the lamp, and is connected with another electroscope. Describe the indications of this electroscope while the machine is in action and the lamp burning, and also after the lamp is blown out.

6. A stick of sealing-wax rubbed with flannel is held over a gold leaf electroscope, and the cap touched for a moment with the finger.

(i) What will be the state of the leaves?

(ii) If the stick be brought nearer the cap, what will be the indication?

(iii) If the stick be moved further away, what will be the indication?

(iv) What will be the effect of holding a large insulated plate of metal between the sealing-wax and the cap? What effect will the thickness of the plate have?

(v) What will be the effect if the sheet of metal be uninsulated?

(vi) What will be the effect of substituting a plate of paraffin for the metal plate?

7. There are two similar gold leaf electroscopes, one with a point attached to the cap. A piece of sealing-wax rubbed with flannel is held over each of them and removed. Describe the indications of the two electroscopes before and after the removal of the sealing-wax.

8. An insulated metal cylinder, positively electrified, is held with its axis vertical, and a funnel whose nozzle projects along the axis of the cylinder to near its middle has water poured into it.

(i) The funnel is uninsulated, determine the electrical state of the issuing water.

(ii) If the funnel now be insulated what effect will be produced on the electrification of the issuing jet at first, and after a time?

(iii) In question (i), after the water has run through, the funnel is insulated and removed, what will be the nature of its electrification? Will it differ from that of the funnel in question (ii) after the water is exhausted?

(iv) What effect will be produced on the issuing jet, by connecting the funnel with the cylinder?

(v) In (i) the issuing stream of water flows into another funnel, which is contained inside a second insulated cylinder and connected with it. What will now be the state of the issuing stream, and what would be the electrical state of the second cylinder supposed neutral at first?

(vi) Will the potential of the lower cylinder go on increasing without limit; or if there be a limit, on what will it depend?

(vii) Show how an arrangement depending on the principle of the preceding questions could be constructed, by which a small charge given to a Leyden jar could be augmented to a high degree.

9. A positively electrified particle repels every other positively electrified particle, but two conductors charged with positive electricity do not necessarily repel each other. Explain this apparent paradox.

10. Show that two equal conductors similarly placed with respect to each other, both raised to the same potential, and insulated, always repel each other.

11. Show that if the potentials of the two conductors in the last question differ ever so little, they will, at great distances, repel each other, but at very near distances (supposing no spark to pass) they will attract each other.

12. Two very thin parallel plates are pressed closely together, insulated and electrified. Show that the work done by them during separation equals half the whole energy of the electrification.

13. Two thin parallel plates are electrified to the same potential, draw a rough diagram of the lines of force.

14. The two thin plates in the preceding question are electrified to slightly different positive potentials. Draw the lines of force, and show that when very near together there will be an attractive force, and when very far apart a repulsive force between them.

Using the notation of Art. 84, the energy of the system is

$$\tfrac{1}{2}C(V_1 - V_2)^2 + \tfrac{1}{2}C'(V_1^2 + V_2^2).$$

Then if C_x, C_x' be the rate of change of the capacities C, C' as x the distance of the plates is increased, the force helping separation will be

$$\tfrac{1}{2}C_x(V_1 - V_2)^2 + \tfrac{1}{2}C_x'(V_1^2 + V_2^2).$$

Now (Art. 94) $C = \dfrac{S}{4\pi x}$, and therefore (Art. 95) $C_x = -\dfrac{S}{4\pi x^2}$. The value of C' we do not know, but it certainly increases with x (Art. 105). Hence for the force separating the plates we have

$$-\frac{S}{4\pi x^2}(V_1 - V_2)^2 + \tfrac{1}{2}C_x'(V_1^2 + V_2^2).$$

This shows that if $V_1 - V_2$ be not zero, the force must certainly be attractive if the plates are near enough.

Again, the first term becomes as small as we please by increasing x and the force will then be repulsive. Also if $V_1 - V_2$ be small the first term will become insignificant even for such moderate values of x that the assumed form of expression for the capacity still remains true. Hence we infer that in bringing the plates very near there will be an attraction between them, and on separating them far enough apart a repulsion.

15. Two spheres of radii 4 and 5 centimetres are connected by a long and fine wire, find the proportion in which a charge communicated to the system is divided between the spheres.

16. A sphere of radius one decimetre is connected by a long wire with a plate one decimetre square, which has at

distance one millimetre from it another parallel plate connected with the earth. Find the ratio in which a charge will be divided between the plate and sphere. Calculate also the numerical capacity of the whole system.

Ans. π to 25; $\dfrac{10(\pi + 25)}{\pi}$.

17. A thin circular plate whose radius is one decimetre is charged with a unit of electricity, and moved till distant one millimetre from a similar plate connected with the earth. Compare the potential of the plate before and after the movement of the plate. *Ans.* 25π to 2 nearly.

18. Two spheres, each one decimetre in radius, are connected by a wire. A third conducting sphere is concentric with and envelopes one of the spheres, and is also connected with the earth: the distance between the surfaces being two millimetres. Show in what proportion a charge communicated to the system is divided. *Ans.* 51 to 1.

19. A Leyden jar one millimetre thick, and having 1 sq. decimetre surface, is fully charged by 5 turns of an electrical machine. How many turns are necessary to charge a battery of 40 square decimetres, and 6 millimetres thick?

Ans. $33\frac{1}{3}$.

20. With same data as ques. 19, what fraction of full charge will be communicated to a battery of 20 sq. decimetres, ·5 mil. thick, by 45 turns of the machine? *Ans.* $\frac{9}{40}$.

21. With same data as ques. 19, a battery having 200 sq. decimetres is charged by 500 turns of the machine. Find the thickness. *Ans.* 2 mm.

22. Compare the energy of discharge of two batteries, one of 20 sq. decimetres, and the other of 80 sq. decimetres, both of same thickness, and charged to same potential.

23. Compare in last question the energy of discharge of the two batteries, supposing one charged with 80, and the other with 150 turns of the machine, neither being supposed fully charged. *Ans.* 256 to 225.

24. Compare the energy of discharge in two batteries, one of 15 sq. decimetres and the other of 60 sq. decimetres, each charged by the same number of turns of the machine, the thickness being the same in both. *Ans.* 4 to 1.

25. Compare the capacities of two batteries, one of 40 sq. decimetres, 1 mil. thick, the other of 100 sq. decimetres, 1·5 mil. thick. *Ans.* 3 to 5.

26. Compare the potentials of two batteries, one of 30 sq. decimetres surface, $1\frac{1}{2}$ mil. thick, the other of 80 sq. decimetres surface, ·8 mil. thick, charged with equal amounts of electricity. *Ans.* 5 to 1.

27. Compare the potentials of the two batteries in the last question, supposing one charged with 10 turns of the machine, and the other with 40 turns, supposing neither fully charged. *Ans.* 5 to 4.

28. Compare the amounts of heat evolved in the discharge of the two batteries of the last question. *Ans.* 5 to 16.

29. A battery of 20 sq. decimetres charged with 40 turns of the electrical machine will just puncture glass ·3 mil. thick. What extent of coated surface of the same thickness, charged to the same potential, will pierce a sheet of glass 3 mil. thick?

30. Two parallel conducting plates are connected, one with the earth, and the other with a source of electricity of constant potential. A positively electrified particle falls from the positive to the negative plate. Show that

(i) The particle falls under a uniform acceleration which varies inversely as the distance of the plates.

(ii) The time of falling is directly proportional to the distance between the plates.

(iii) The velocity acquired in falling is independent of the distance.

31. A gold leaf electroscope is connected by a long wire with various points in succession on an electrified conductor, the distribution being (1) free, (2) induced. What difference (if any) will there be in its indications?

32. In what respects will the indications of the preceding question differ (1) from those obtained by touching the various points with a proof plane and bringing it near the electroscope, (2) from the results obtained by suspending pith balls at various points on the conductor?

33. The plates of a condensing electroscope are connected by a long fine wire, electrified and separated. Will there be any change observed in the divergence of the leaves during separation?

34. How are the potentials of the surfaces of a charged and insulated Leyden jar affected by letting down into it a conductor—

(i) Connected with the earth?

(ii) Completely insulated?

(iii) Completely insulated, but left with one half extending outside the jar?

35. Two plates, having gold leaves attached to their faces, are charged as a Leyden jar, and insulated. The distance between the plates is now varied. Discuss fully the changes in the state of the gold leaves as the distance is varied.

36. A Leyden jar is charged and placed on the cap of a gold leaf electroscope. A small body, neutral or electrified, is brought near the knob of the jar and then removed. Describe and explain all the indications of the electroscope (1) when the body is neutral, (2) when positive, (3) when negative.

37. What differences would there be in the preceding question, if the body be allowed to touch the knob of the jar?

38. A series of n jars, whose capacities are C_1, C_2, C_3,... are charged by cascade and fitted up as a battery. Show that if we neglect the free charges, and V denote the potential of the source, and Q the charge of the battery,

$$\frac{nV}{Q} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots\dots\dots$$

39. Hence show that the charge of a battery charged by cascade is greater than that of the jar of lowest and less than that of the jar of highest capacity, charged from the same source.

40. Three jars are connected by fine wires and charged by cascade. Show how to calculate the electrification of the system, making allowance for the free charge.

Ans. If C_1, C_2, C_3 be the capacities for bound charge,

i_1, i_2, i_3 be free capacities of charges on inner coats,

o_1, o_2, o_3 be free capacities of charges on outer coats,

and V be potential of source, the potentials of the inner coats of the second and third jars are respectively

$$\frac{C_1(C_2+C_3+i_3+o_2)}{(C_2+C_1+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}\cdot V,$$

$$\frac{C_1C_2}{(C_2+C_1+i_2+o_1)(C_3+C_2+i_3+o_2)-C_2^2}\cdot V.$$

41. A number of conductors of capacities C_1, C_2,... are raised to potentials V_1, V_2,... respectively, and afterwards connected by fine wires so as to form one conductor. Show that the potential of the conductor is given by

$$\frac{\Sigma CV}{\Sigma C}.$$

42. Show that in the preceding system the energy of the whole conductor is to the energy of the separate conductors as $\{\Sigma(CV)\}^2$ is to $\Sigma C\times\Sigma(CV^2)$.

43. Two equal jars are charged one positively and one negatively to the same potentials. The inner and outer coats are then connected by wires. What will be the state of each jar?

44. In the preceding question, does electricity pass from one outer coat to the other when both are at zero potential?

45. In the last question but one, what would happen if the outer coats were insulated and the inner coats connected by a wire?

46. Two insulated spheres, whose radii are r_1 and r_2, are in regions of potentials V_1 and V_2 and are connected by a fine wire:

(i) Find the potential of the system.

(ii) Find the free charges, supposing the wire to be suddenly cut by a pair of scissors with glass handles.

47. A ball is insulated and held within a Leyden jar, being connected by a wire with an electroscope outside.

(i) What will be the indication of the electroscope when the jar is first charged?

(ii) What would be the simultaneous indication of another electroscope entirely within the Leyden jar and connected with the ball?

(iii) If after charging the jar the ball be touched by the finger, what will be the indication of the outside electroscope?

(iv) How will the indication of the electroscope be affected if the jar gradually leak?

(v) If the Leyden jar be charged and insulated before the introduction of the ball, how will the introduction of the ball affect an electroscope connected with its outer surface?

48. A plate of radius a has another plate of the same radius at a small distance T from it.

(i) If the system be charged as a Leyden jar, compare the free and bound charges. *Ans.* As $4T$ to $a\pi$.

(ii) If the plates of the Leyden jar be insulated and removed to a distance t, find the potential of the plates and the amount of the free and bound charges. Discuss both the cases in which t is greater and less than T.

(iii) The plate receives a charge Q of electricity and is moved till it is distant T from the second plate, which is connected with the earth. Find the potential and the amounts of the free and bound charges.

Ans. Potential $= \dfrac{\pi}{a} \cdot \dfrac{4T}{4T + a\pi} \cdot Q.$

(iv) If $a = 1$ decimetre and $T = {\cdot}01$ cm. and the system be charged as a Leyden jar, calculate the rise in potential of each plate when the plates are entirely separated. How is this applied in the condensing electroscope?

Ans. Ratio of 1 to 125π.

(v) The two plates are charged as a Leyden jar and the positive plate is removed, the negative plate being left uninsulated. Find the whole work done. (See Art. 105.)

49. A Leyden jar is charged in the usual manner and insulated. The knob is now touched by the finger. Find the change in potential of the two coats and calculate the energy of the discharge.

50. Two Leyden jars charged to different potentials have their knobs brought for an instant into contact. Calculate the energy of the spark which passes.

51. A Leyden jar is charged, and the charge divided with another equal Leyden jar, which is uncharged. Show that one-half the whole energy of the system runs down in the spark.

52. Find the energy expended in charging a conductor of known capacity to a given potential by means of a unit jar, whose potential of discharge is known. Show that it will be independent of the capacity of the unit jar.

53. A Leyden jar is charged and fitted up with $(n-1)$ uncharged similar jars to form a battery, show that the whole energy is only $\frac{1}{n}$ of the energy of the single jar.

54. A plate is placed between two equal and parallel plates, and the three are electrified to given potentials, the middle plate being highest. Find the position of equilibrium of the middle plate.

55. If the middle plate, when in a position of equilibrium, be removed, find the amount of its charge.

56. Show that the equilibrium in the preceding example is unstable.

57. Explain the necessity of a bifilar or analogous suspension in the needle of a quadrant electrometer.

58. Explain the experiment of 'Mahomed's Coffin.' A small chip of gold leaf, pointed at one end and blunt at the other, is thrown into the air near the knob of a charged Leyden jar, and is observed to remain freely suspended for some time.

59. Show how to find the potential at any point between two parallel plates electrified to different potentials.

60. A small sphere is insulated and placed between the parallel plates in the last question, show how to determine its potential.

61. If a small insulated sphere be placed between two concentric spheres, charged as a Leyden jar, show how to find its potential.

62. Show that any symmetrical conductor, placed symmetrically in a uniform field of force, will have the same potential as that at its centre, supposing the conductor removed.

63. A sphere of radius r is insulated in a large uncharged Leyden jar without contact with the walls. It is connected by a long wire with another sphere of radius R, insulated in a region of zero potential. The jar is charged, and its potential rises uniformly at the rate of v units per second; find the rate of flow of electricity through the wire.

Ans. $\dfrac{rR}{r+R}\,v$ units per $1''$.

64. Deduce the rate of flow in the preceding example, supposing the wire to have its distant end to earth.

65. A soap-bubble is blown and afterwards electrified. Find an expression for the radius of the soap-bubble that the internal pressure on the soap-film may be constant as the electrification proceeds.

Ans. $\rho^2 = \dfrac{\Pi}{2\pi}\left(1 - \dfrac{a^3}{r^3}\right)$, where Π is the constant pressure, a the initial radius, r, ρ the radius and electrical density at any time during electrification.

66. In *Holtz's Machine* or in the *electrophorus*, any amount of electricity however small is made to produce an amount of electrical separation as great as we please. Can this be reconciled with the principle of conservation of energy?

67. An unelectrified conductor at zero potential on being insulated and introduced into a space at potential V assumes the potential of the space. Show how this can be reconciled with the principle of conservation of energy.

68. Point out how the same principle is satisfied in the water-dropping apparatus described in ques. 8.

69. Show that in a system of equipotential surfaces round an electrified sphere, the distances of the consecutive members of the system of equipotential surfaces from the centre of the sphere form an Harmonical Progression.

70. If any system of equipotential surfaces be freely electrified, the capacity of any surface varies inversely as its potential, supposing each surface to enclose the whole electrical system.

71. Show that the rate of movement of any equipotential surface as the electrification proceeds at a uniform rate, varies inversely as the product of the force at the point on the surface multiplied by the capacity of the surface for a free electrification.

72. Show that in the case of an electrified sphere, the rate of electrification is equal to the velocity of any equipotential surface multiplied by its potential.

73. If a sphere be at zero potential, and have its centre at a distance f ($>$ radius) from a particle having m units of electricity, show that the quantity of electricity on the sphere is $-\dfrac{am}{f}$, a being the radius.

74. If the sphere be charged with Q units of electricity, and brought near a point having m units of electricity, find the potential within the sphere.

Ans. $\dfrac{Q}{a}+\dfrac{m}{f}$.

75. If a hollow sphere be charged with Q units of electricity, and have a particle charged with $-q$ units introduced through a small aperture, find the position of the particle that the potential at the centre may be zero.

Ans. Distance from centre $= \dfrac{aq}{Q}$.

76. A sphere near an electric system is brought to zero potential and insulated. On being removed the potential of the sphere is found to be $-V$. Show that the sphere occupied such a position that the potential at its centre due to the given system was $+V$.

77. Two spheres of unequal radii are charged to the same potential, insulated, and brought near to each other till a spark passes. Find in which direction the spark will pass between the spheres.

78. To find the work done in moving a particle charged with a given quantity of electricity from any given point within a sphere to its centre.

Let P be the position of the particle charged with m units of electricity and CPT a diameter, T being conjugate to P. Let P', T' be a pair of conjugate points on the same diameter near to P, T.

The force on m at P is only that due to attraction of $-\dfrac{a}{f}m$ at T;

$$\therefore \text{ Force} = \frac{m \cdot \dfrac{a}{f} m}{PT^2} = \frac{am^2 f}{(a^2 - f^2)^2}.$$

But if PQ, $P'Q'$ be drawn perpendicular to CPT

$$a^2 - f^2 = CQ^2 - CP^2 = PQ^2;$$

$$\therefore \text{ Force} = \frac{am^2 . CP}{PQ^4}.$$

Hence average force over $PP' = \dfrac{am^2}{2} \cdot \dfrac{CP + CP'}{PQ^2 . P'Q'^2}$.

$\therefore$ Work done through $PP' = \dfrac{am^2}{2} \cdot \dfrac{(CP + CP')(CP - CP')}{PQ^2 \times P'Q'^2}$

$$= \frac{am^2}{2} \cdot \frac{CP^2 - CP'^2}{PQ^2 . P'Q'^2} = \frac{am^2}{2} \cdot \frac{P'Q'^2 - PQ^2}{PQ^2 . P'Q'^2}$$

$$= \frac{am^2}{2} \left(\frac{1}{PQ^2} - \frac{1}{P'Q'^2} \right).$$

Adding the whole work from a given point K to the centre

$$= \frac{am^2}{2} \left(\frac{1}{CA^2 - CK^2} - \frac{1}{CA^2} \right) = \frac{f^2 m^2}{2a(a^2 - f^2)},$$

supposing $CK = f$.

79. A sphere is at zero potential, find the work done in removing a particle charged with a given quantity of electricity from any external point to an infinite distance.

Ans. $\dfrac{am^2}{2(f^2 - a^2)}$.

80. A sphere is charged with a given quantity of electricity, find the work done in moving a particle from any given external point to an infinite distance.

Ans. $\dfrac{am^2}{2(f^2 - a^2)} - \dfrac{Qm}{2f^2}(2f + a)$, Q being the given charge.

81. A very large insulated circular plate has a particle charged with m units of electricity very near to its centre, find the potential of the plate.

Ans. $\dfrac{m\pi}{2a}$ nearly.

82. A sphere having a charge of electricity is brought near an electrified particle. Find an expression for the density of the electrification at any given point.

Ans. Let Q be the charge of the sphere, a its radius, q the charge of the electrified particle, f its distance from the centre; then the density at a point distant r from the electrified particle is

$$-\frac{q(f^2 - a^2)}{4\pi a r^3} + \frac{Q + \dfrac{a}{f} m}{4\pi a^2}.$$

83. An insulated, but unelectrified sphere, is brought near an electrified particle. Find the position of the line of neutral electrification.

Ans. Using the same notation as in ques. 82, the distance from the electrified particle is

$$\sqrt[3]{f(f^2-a^2)}.$$

84. A uniformly electrified ring is placed in a diametral plane of a sphere at zero potential and is concentric with it, find the density of the charge at either pole.

85. If a ring, having the same radius as a sphere, be placed in a tangent plane to the sphere, so that the point of contact is the centre of the ring, compare the electrical density at the centre of the ring and at the opposite extremity of the diameter, the potential of the sphere being zero.

86. Given the amount of electrification of the ring, find the amount of the whole induced charge in each of the two last questions.

87. If the ring be placed in a symmetrical manner inside the sphere, find the density at the two poles.

88. A closed region, whose surface is a bad conductor, encloses a very delicate electrometer; electrified bodies are moving about with great velocity outside the closed region, will the electrometer give any indication?

89. What would be the best form of electrometer for conducting the above experiment, and in what manner would you fit it up to make the indications as great as possible?

90. If the movement were one of rotation round the closed space, so as to keep the moving bodies on the whole at a constant distance, would there be any indications? How would an observer, placed outside the region, proceed to make observations in this case?

91. If you were in a closed space, having only a small aperture, how would you proceed to determine the electrification of the space?

92. How far does the method you employ in the preceding question apply to determine the *absolute* electrification of the earth?

93. What would be the electrical state of a sky-rocket just before reaching the earth?

94. A balloon is allowed to ascend from the earth carrying a burning match, which is kept connected with one terminal of a quadrant electrometer by means of a fine insulated wire, which is let out as the balloon ascends. During the first hundred yards the potential rises gradually at the rate of 1° per 10 yards of ascent. After this the register is constant for 20 yards, for the next 50 yards it falls at the rate of 1° per 15 yards, and again rises uniformly at the rate of 1° per 12 yards of ascent. What inferences as to atmospheric electricity would be drawn from these observations?

95. Explain why in a Leyden jar the loss of charge appears more rapid a few minutes after first charging than it does afterwards.

96. A Leyden jar is charged and left for a few minutes, when its charge is divided by instantaneous contact with another equal jar. State what will be the condition of the two jars a few minutes afterwards.

97. A Leyden jar made of a plate of shellac, coated on both sides, is charged, discharged and the coats removed. What will be the electric state of the surface of the shellac, and how will it vary with time?

98. If two spheres, placed in oil of turpentine, be charged to given potentials, will the force between them be greater or less than in air?

99. If two spheres be charged with given quantities of electricity and placed in oil of turpentine, will the force between them be greater or less than in air?

100. A metal sheet is placed between two plates of non-conducting matter, whose inductive capacities are K and K', and their thicknesses t and t', and two other metal sheets are placed outside the plates. The inner sheet is kept at potential V, while the outer sheets are at zero. Compare the charges on the outer sheets on being insulated and removed.

101. Faraday constructed a room coated externally with tinfoil and furnished with an aperture or window. The whole

room was insulated on glass legs, and could be powerfully charged by a large frictional machine.

(i) On charging the room, what effect would be produced on electrometers placed inside it?

(ii) How would a person inside proceed to determine the external electrification of the room?

(iii) If a frictional machine be carried inside the room and worked, the rubber being connected with the walls of the room, how will a gold leaf electroscope, placed outside in contact with the external surface, be affected?

(iv) If a ball be charged inside the room, insulated and carried out, what effect will be produced on the electroscope?

(v) A number of conductors are charged from the machine within the room, and suspended by silk threads within the room, how will these affect the external electroscope?

102. A ball is electrified and held above a metal plate, which is then touched by the finger, what indications would be obtained by testing the plate at various points above and below with a proof plane?

103. A metallic ball is lifted by a silk fibre on to the top of a rod of sealing-wax, the lower part of which has been rubbed with a silk handkerchief, what indications would be obtained by touching it at various points with a proof plane?

104. What differences would there be in the last question if the ball had been placed on the sealing-wax by hand?

105. Two large spaces are constructed, which are kept at constant potential, one A at potential V_1, the other B at potential V_2, supposing $V_1 > V_2$. Two spheres of equal radii are placed in these regions insulated from them, and connected by a fine wire also insulated.

(i) What will be the potential and the amount of charge on each sphere?

(ii) What would be the indication of an electroscope placed in space A, and connected with its sphere?

(iii) What would be the indication of an electroscope placed in space B, and connected with its sphere?

(iv) A burning metal lamp is placed on the sphere in region A, how will the indications of the two electroscopes be affected?

(v) If a burning metal lamp be placed on each sphere, how will the indications be affected?

(vi) What will be the effect on the indications of the electroscopes if the wire be at some point to earth?

106. A sphere of radius one centimetre is charged with a unit of electricity and placed in a space at potential 10, what will be the potential of the sphere?

107. A sphere of radius unity is introduced into a place at potential 5, and then connected with the earth. What will be its free charge on being insulated and removed?

108. A conductor whose capacity is 4, is introduced into a room whose potential is 4, and the conductor is then brought to potential 3, insulated, and removed. What will be the amount of the electrification?

109. A conductor whose capacity is 6, is charged with 12 units of $-$ electricity, and placed in a region at potential 3. What will be the potential of the conductor?

110. A conductor at zero potential is in a space at potential 8; on being insulated and removed it has 24 units of $-$ electricity. What is its capacity?

111. A conductor of capacity C is charged with Q units of electricity, and put in a space at potential V. What will be the potential of the conductor?

112. A conductor is brought to zero potential in a space at potential V. On being insulated and removed it is found to have $-Q$ units of electricity. What is its capacity?

113. A conductor of capacity C is placed in a region at potential V, and brought to potential V'. Find its charge.

114. If the prime and negative conductors of an electrical machine have equal capacities, show that the effective working of the machine is at first diminished by one half, when the negative conductor is insulated.

115. If the capacities in the last question are in the ratio C_1 to C_2, find the ratio in which the effective working is at first diminished by insulating the negative conductor.

116. If an electrical machine be placed in the open air at a height h from the earth, and worked (with rubber uninsulated) till the prime conductor has a charge e of electricity, when the earth connection is broken; show that negative electricity is spread over the earth with a density at any point represented by $\frac{he}{2\pi r^3}$, where r is the distance of the point from the machine.

117. Show also that the change produced in the potential of the earth is to the potential of the conductor as $-hC$ to R^2, where C is the capacity of the conductor and R the radius of the earth.

118. By inverting the electrification of a circular disc with respect to its centre, find the electrification of an infinite plate connected with the earth, having a circular aperture, under the influence of an electrified particle at the centre of the aperture.

Ans. If e be the quantity of electricity at the point, and a the radius of the aperture, the density at a distance r from the centre

$$= -\frac{1}{2\pi^2} \cdot \frac{ae}{r^2 \sqrt{r^2 - a^2}}.$$

119. By inverting the electrification of a circular disc with respect to any point in a line perpendicular to it through its centre, find the electrification of a bowl in the shape of a spherical segment, having an influencing particle at its opposite pole.

Ans. If e be the quantity of electricity at O, the influencing point, P the point on the bowl, A, A' the points in which a diametral section through OP cuts the rim, then the density at P

$$= -\frac{1}{2\pi^2} \cdot \frac{e}{OP^2} \cdot \frac{OA}{\sqrt{PA \cdot PA'}}.$$

CHAPTER V.

THEORY OF THE VOLTAIC CELL.

137. We have stated that when two conductors brought by means of an electrical machine to different potentials are joined by a conducting bridge, an equalization of potential takes place through the bridge, which we may represent as a flow of electricity from the place of higher to that of lower potential, or briefly as a current of electricity. We have, moreover, calculated the mechanical equivalent of such a discharge, the energy being converted into heat in the bridge, or into work external to the bridge in a variety of ways. The phenomena belonging to the bridge while the current is passing form the special subject for consideration in Voltaic Electricity or Galvanism.

The current obtained by means of the common form of friction-machine is a single instantaneous discharge, or a rapid succession of such instantaneous discharges, and therefore ill adapted for the production of the class of phenomena to which we have alluded. They can be observed to perfection by means of the galvanic battery, in which the electrical separation takes place with such rapidity, that the successive discharges, if they exist, cannot be separated by the most delicate tests. We must bear in mind, however, that the differences of potential with which we are concerned are extremely minute compared with those obtained in the machine, while the quantity of electricity in motion is incomparably greater. To return to our old hydrostatical analogy, the machine current is a tiny stream tumbling down a precipitous hill-side, the galvanic current is a vast lake flowing through an almost level valley.

138. Before proceeding to the phenomena themselves, we shall consider the connection between the two modes of generating electricity.

In all frictional electrical machines the source of electricity is ultimately the friction of two bodies of different substances, which, when rubbed together, appear to exercise an unequal attraction for the opposite electricities, which were at first neutral in both bodies. The result of this unequal attraction is the production of a difference of potential between the bodies, this difference, while they are in contact, depending on the nature of the rubbing surfaces, and on the amount of rubbing.

The energy represented by this difference of potential is derived from the mechanical rubbing, as also are the heat and change in character of the two surfaces which accompany it.

For the development of the current, it appears necessary that there should be at least three heterogeneous bodies arranged in a circuit, one of such bodies, at least, capable under some conditions of being decomposed and forming a chemical compound with one of the other two.

139. Suppose A, B, C to be three such bodies, of which A, B exercise a chemical affinity for each other. The development of the current has been attributed to one of two causes:—

(i) To the differences of potential produced at the three places of contact, A with B, B with C, and C with A. This is Volta's or the Contact Theory.

(ii) To the chemical attraction between A and B, which throws the circuit into a state of polarization; the resulting chemical action being accompanied by an electrical discharge round the circuit; the current being the result of a rapid succession of such alternate polarizations and discharges. This is Faraday's or the Chemical Theory.

These two theories of the action of the cell have been warmly debated among physicists, our countrymen for the most part siding with the more recent theory of Faraday, while continental physicists have for the most part accepted the older theory of Volta, though somewhat modified. The point of dispute amounts briefly to this: Volta recognized

that one of the three substances in the circuit must be a fluid; Faraday, however, seeing that the chemical composition of this fluid was, in all cases, altered by the passage of the current, attributed the current solely to this chemical action. As a crucial experiment he constructed a cell in which were two metals and one fluid, the fluid being (for fluids) a good conductor, but not capable of acting chemically on either of the metals. He showed by the most delicate tests, that in this case there was no current in the cell. This, in his opinion, entirely overthrew Volta's Theory. More recently, however, the perception of the law of Conservation of Energy, first put forth by Helmholtz, has shown that in the crucial experiment relied on by Faraday the existence of a current would have been an independent creation of energy. This has again opened the question, and experimenters have diligently set themselves to work to put the theory of Volta again to the test of exact experiment.

So great, however, is the intrinsic difficulty of these experiments, that it is hardly too much to say that at present in no single instance has a difference of potential been directly shown between two bodies, independent of the gaseous medium between them, of the pressure with which they are brought together, and of the friction with which they are separated; the existence of such a difference of potential in every case lying at the very foundation of Volta's theory.

140. Nearly all the experiments hitherto made on the difference of potential caused by contact of two different substances, depend on the principle of the condenser with air as dielectric between the condenser plates. Thus to find the difference of potential in absolute measure between zinc and copper, plates of these metals ground quite true are placed at a measured distance apart, and connected with the terminals of a quadrant electrometer. After connecting the two plates outside the condenser for a moment, the condenser plates are separated and the deflection of the electrometer observed thus giving a direct measure of the difference of potential in question. Experimenting on this principle, the differences of potential at the successive heterogeneous contacts both in a zinc-copper and a Daniell's cell have been given in

absolute measure. A little consideration however will show that in these experiments what is really measured is not the difference of potential between zinc and copper, but that between air in contact with zinc and air in contact with copper, the zinc and copper being in contact; and it has therefore been assumed that a metal is at the same potential as the air in contact with it. That this is not a necessary property of gases is proved by Mr J. Brown (*Phil. Mag.* Aug. 1878), who has shown that copper is negative with respect to iron in air, but is positive with respect to iron in hydrogen-sulphide. The only method depending on any other principle than that of the condenser plate is thus explained by Prof. Clerk Maxwell. (*The Electrician*, April 26, 1879.)

"If we cause an electric current to pass from copper to zinc, the heat generated in the conductor per unit of electricity is a measure of the work done by the current per unit of electricity, for no chemical or other change is effected. Part of this heat arises from the work done in overcoming ordinary resistance within the copper and the zinc. This part may be diminished indefinitely by letting the electricity pass very slowly. The remainder of the heat arises from the work done in overcoming the electromotive force from the zinc to the copper, and the amount of this heat per unit of electricity is a measure of this electromotive force. Now, it is found by thermoelectric experiments that this electromotive force is exceedingly small at ordinary temperatures, being less than a microvolt, and that it is from zinc to copper." The microvolt here alluded to means the millionth part of a volt. Experiments conducted with great care by Profs. Ayrton and Perry give for the same potential difference when estimated on the condenser principle, three-quarters of a volt. These latter experiments are of interest, since they show that the sum of all the potential differences estimated by this method of the different heterogeneous contacts equals the total potential difference between the terminals.

141. To understand how the contact of two substances may produce a difference of potential, we must make some assumptions with respect to the molecular physics of bodies. The assumption usually made is that all bodies have their molecules in a constant state of vibration, while the ampli-

tudes and periods of vibration are different in different bodies.

Thus when the molecules of two different bodies impinge on each other, as at the surface of contact, they cannot accommodate themselves to each other's motion, but constrain each other, this constraint producing a loss of energy. If, however, the two substances are of the same kind and at the same temperature, the molecules on each side of the surface of contact are swinging in exactly the same manner, and can easily accommodate themselves to each other's motion without more constraint than exists in the interior of either body. It is this loss of energy owing to the unsymmetrical swinging of the molecules at the surface of contact which reappears as difference of potential between the two bodies, or as the energy of electrical separation.

The opposed electricities so separated will, for the most part, be heaped up on either side of the plane of separation by a Leyden jar action.

Let A be the area in contact in any particular instance,

Q the quantity of electricity separated,

V the difference of potential produced.

Then the energy of electrical separation is (Art. 78) $\frac{1}{2}QV$.

The molecular energy abstracted is proportional to the area in contact, and may be written mA, where m is a constant depending on the nature of the two surfaces.

Hence $$mA = \tfrac{1}{2}QV.$$

Again, in a Leyden jar of given substance and thickness (Art. 94), the quantity of the accumulation is proportional jointly to the difference of potential and to the area of the surface of the jar. Hence we may write

$$Q = nA \, . \, V,$$

where n is another constant, depending only on the nature of the two bodies. Hence we have

$$mA = \tfrac{1}{2}nA \, . \, V^2,$$

$$\text{or } V^2 = \frac{2m}{n}.$$

Hence V or the difference of potential produced by contact is independent of the shape of the bodies and of the area in contact, depending only on the substances concerned.

We have neglected here the small portion of the electrification which will distribute itself over the two bodies according to electrostatic laws, maintaining the two bodies at a constant potential throughout their mass. This will in all cases be exceedingly small, corresponding to the free charge in a Leyden jar.

142. Suppose now two bodies AB, BC to be joined at one end B. In virtue of the contact one of them, suppose

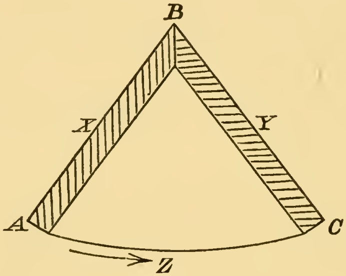

Fig. 47.

AB, acquires a higher potential than the other, BC. If now we could join A and C by a body which behaved *only* as a conductor, a flow of electricity would take place between A and C tending to equalize their potentials; the contact at B would develope a fresh difference of potential, and we should have a continuous current through AC. This current would be a source of energy, and we should, in this case, have an unfailing source of energy. The law of conservation of energy shows this to be impossible. Thus we learn that the contact of C, with A on one side and with B on the other, must produce differences of potential whose aggregate effect is to counteract the difference at the junction B. Or calling X, Y, Z the three bodies, and denoting by $Y \mid X$ the difference of potential between X and Y, assuming X to be at higher potential than Y; we have

$$Y \mid X = Y \mid Z + Z \mid X.$$

If we regard $X \mid Y$ as symbolically equal to $-Y \mid X$ we may write this relation

$$Y \mid Z + Z \mid X + X \mid Y = 0,$$

which must be regarded as a fundamental relation in the case of all bodies whose molecular condition remains unaltered by contact. It expresses the fact that if any number of such bodies be in continuous circuit the difference of potential between the extreme pair is the same as if these two were in direct contact. This was proved experimentally by Volta by means of his condensing electroscope for all metals.

143. In the typical voltaic cell we have two solids, say zinc and platinum immersed in a liquid, say hydrogen chloride, which is capable of entering into chemical combination with the zinc. The relation noted above will not therefore hold, since there will be an alteration in the molecular condition of two of the substances involved.

On dipping the zinc plate into the fluid, a difference of potential Zn | HCl is established between them, and on dipping the platinum plate in, a difference Pt | HCl is established. The fluid being a conductor, a distribution of electricity over its surfaces takes place instantaneously, and establishes equality of potential throughout the fluid mass. The zinc and platinum plates are therefore at different potentials, the amount of difference being

$$\mathrm{Zn} \mid \mathrm{HCl} + \mathrm{HCl} \mid \mathrm{Pt}.$$

This difference could be tested by a quadrant electrometer, provided the alternate pairs of quadrants were of zinc and platinum respectively.

Suppose now a zinc wire laid across from the zinc to the platinum plate. At the point of contact with the platinum a new difference of potential is introduced represented by Pt | Zn.

The whole difference of potential between the zinc plate and the other end of the zinc wire then becomes

$$\mathrm{Zn} \mid \mathrm{HCl} + \mathrm{HCl} \mid \mathrm{Pt} + \mathrm{Pt} \mid \mathrm{Zn}.$$

If the three substances followed Volta's law, this would necessarily vanish. Since however hydrogen chloride has

chemical affinity for the zinc, it will not vanish, and the ends of the wire being now at different potentials, a flow of electricity takes place *through the wire* from the platinum towards the zinc plate, tending to equalize their potentials.

144. In consequence of this, the fluid in contact with the zinc acquires a higher potential than that in contact with

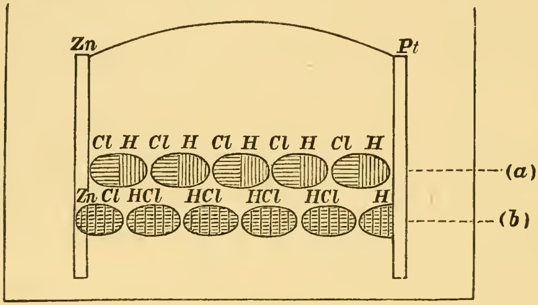

Fig. 48.

the platinum. The molecules of the fluid become polarized, having their positive ends turned towards the platinum, and their negative ends towards the zinc. Hydrogen chloride being a compound body, *we assume* that the elements hydrogen (H) and chlorine (Cl) exercise their own electrical affinities, the hydrogen being the electro-positive, and chlorine the electro-negative component. The arrangement of the compound molecules might be shown thus (fig. 48, *a*).

The chemical affinity of the zinc and chlorine now comes into play, causing the Zn to combine with the chlorine atoms next to it, so as to form zinc chloride ($Zn\,Cl_2$, two atoms of chlorine combining with each atom of zinc). The hydrogen of this molecule combines with the chlorine of the next, and so on along the whole row of molecules, leaving the hydrogen free at the platinum plate, the molecules at the same time each becoming neutral. This arrangement is shown in *b*, fig. 48. In this way the discharge of electricity has travelled round the whole circuit. The platinum plate is again brought to a higher potential than the zinc, and the

same process is repeated, the successive discharges following each other with so great rapidity that their existence can only be inferred from theoretical considerations*.

We find, however, apart from all theory, after the passage of the current for any length of time, that zinc is consumed, zinc chloride is formed in the cell, and hydrogen bubbles up at the platinum plate. So far our provisional theory accounts for the facts observed.

We find moreover, that during the passage of the current, heat is developed in all parts of the circuit, and that the conductor is capable of performing work external to itself (as the movement of a magnetic pole, for instance). We are in consequence compelled to look for a source of energy in the circuit. This source we find in the combination of zinc and chlorine. Whenever zinc chloride is formed, heat is evolved in the process, and it is found by actual experiment that the whole heat evolved (supposing no other work done) during the passage of the current is the same as that which would be given out by dissolving in Hydrogen chloride the amount of zinc that has combined with chlorine in the cell.

145. The electro-chemical property of decomposable fluids noted above has been explained by saying that a metal in contact with a fluid exercises not only a *mass attraction*, but also an *atomic attraction*. The difference of potential between zinc and hydrogen chloride may be resolved into Zn | HCl the mass attraction, and [Zn | HCl] or [Zn | H + Zn | Cl] due to the attraction of the zinc for the separate atoms, the latter being denoted by being in-

* The dissociation theory of Clausius is now considered preferable to the polarization theory of Grotthüs given in the text. According to Clausius the liquid HCl always contains a proportion of molecules which are in the act of breaking up into H and Cl; while the free atoms moving rapidly through the liquid become associated into new molecules of HCl. During the periods in which the atoms are free, the electro-positive H atoms drift towards the platinum plate while the electro-negative Cl atoms drift towards the zinc. The net result is a constant flow of H atoms giving up positive electricity to the platinum and of Cl atoms giving negative electricity to the zinc: thus completing the electrical circuit.

cluded in brackets. We might then write the whole difference of potential

$$\mathrm{Zn} \mid \mathrm{HCl} + [\mathrm{Zn} \mid \mathrm{HCl}] + \mathrm{HCl} \mid \mathrm{Pt} + [\mathrm{HCl} \mid \mathrm{Pt}] + \mathrm{Pt}\ \mathrm{Zn}.$$

We may now assume that, as far as the mass attractions are concerned, the substances obey Volta's law, so that

$$\mathrm{Zn} \mid \mathrm{HCl} + \mathrm{HCl} \mid \mathrm{Pt} + \mathrm{Pt} \mid \mathrm{Zn} = 0,$$

and the unbalanced difference of potential which originates the current is

$$[\mathrm{Zn} \mid \mathrm{HCl}] + [\mathrm{HCl} \mid \mathrm{Pt}],$$

due only to the atomic attraction of the metals on the elements of the fluid.

146. We must now define three terms of constant use in reference to a voltaic cell.

DEF. ELECTROMOTIVE FORCE *is used to denote the sum of all the differences of potential effective in a voltaic circuit.*

The term electromotive force is convenient, as we apply it to all cases in which a current is originated, even when we cannot strictly say that there is a difference of potential. It should also be noted that it is not a force in Newton's sense of the word, but Potential energy per unit of electricity.

DEF. ELECTRODES. *The metal plates which dip into the fluid are called* electrodes, *that to which the external current flows being the* zincode, *and that from which it flows the* platinode. *The term is also extended to any two terminals from and to which electricity flows.*

DEF. POLES. *The term* pole *is used of the extremities of the conductor external to the fluid, that in connection with the platinode being the* positive pole, *that in connection with the zincode the* negative pole.

The direction of the current will therefore be in the fluid from the zincode to the platinode, and external to the fluid from the positive to the negative pole.

147. It is sometimes convenient to represent graphically the changes of potential in the course of a circuit. When the circuit is open this can easily be done provided we know

in absolute measure the value of the successive differences that occur.

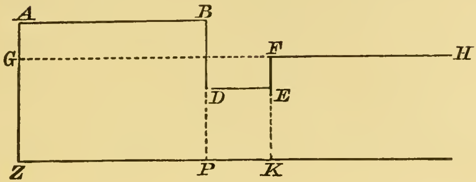

Fig. 49.

Thus, in the typical cell, let Z be the zinc plate, P the platinum plate, and K the junction of the zinc wire and platinum plate.

We *assume* the differences of potential to be

at Z, Zn | HCl, which shall be positive, and may be represented by ZA;

at P, HCl | Pt, which shall be negative and less than ZA, let it be BD;

at K, Pt | Zn, which shall be positive and equal to EF.

The broken line $ABDEFH$ gives us the law of change of potential throughout the circuit. The whole electromotive force of the cell is represented by

$$ZA - BD + EF = ZG \text{ suppose,}$$

and this would be the difference of potential of the two terminals or poles, as measured by a quadrant electrometer.

148. We say nothing here about the potential at any part of the circuit, which, if the cell be insulated, will be positive at one terminal and negative at the other. In practice, one part of the circuit is generally put to earth, and thus brought to zero potential. If the zinc plate be put to earth, then the ordinates in the figure represent the potentials, and ZG is the potential at the other end of the open circuit.

If the platinum plate were put to earth the potentials would be shown by ordinates drawn to a horizontal line through DE, that at Z being equal to $-DP$ or $-EK$, and that at H to EF, the whole difference being in all cases equal to ZG.

149. The electromotive force of any cell can now be calculated *à priori*, if we know the differences of potential produced at the various contacts. The experimental difficulties render these determinations very unreliable, and we consequently content ourselves with knowing the whole electromotive force active in the cell, which is the only thing that practically concerns us. Having determined this for one cell in absolute measure, we can compare the electromotive forces of different cells with it, by methods to be explained further on.

150. It is well known that metals possess a remarkable power of condensing gases on their surface, and the electrical influence of these gases is seen in a variety of ways.

If two platinum plates be placed in hydrogen and oxygen gas respectively for some time, and be afterwards dipped in water (slightly acidulated to improve conduction), and then joined by a wire, a current is found to pass from the oxygen to the hydrogen plate. Since there is no contact of heterogeneous substances except platinum and water (which occurring twice, the differences should neutralize each other), the electromotive force must be due to a difference in behaviour towards water of a plate charged with oxygen and one charged with hydrogen. In this cell the electromotive force may be represented by

$$Pt_H \mid H_2O + H_2O \mid Pt_O,$$

where Pt_H and Pt_O denote respectively that the plate is charged with hydrogen and oxygen. The passage of the current is accompanied by the disappearance of the free gases, which recombine to form water, and the cell is therefore active only for a short time. The energy of the cell was abstracted from the kinetic energy of the gases, and is equal to the energy of chemical separation of oxygen and hydrogen.

The same effect arises in all cells in which gas is liberated at the positive plate, unless the gas be soluble in the liquid round the plate. The liberated gas causes a backwards electromotive force which diminishes the effective electromotive force in the cell, and weakens the cell as soon as it is in action. This effect is commonly known as polarization.

151. To avoid this, a variety of cells have been constructed, in which the substance liberated at the positive plate is not gaseous, or if so, a gas which is soluble in the liquid which surrounds it.

In Daniell's cell there are two compartments divided by a membrane or a porous diaphragm, through which transmission of fluid and chemical action takes place. In one compartment is placed a zinc rod, immersed in dilute sulphuric acid (H_2SO_4), and in the other a rod of copper immersed in copper sulphate ($CuSO_4$). In this cell the radical SO_4 (sulphion) takes the place of chlorine in the former cell, zinc sulphate being formed at the zinc plate, hydrogen sulphate at the diaphragm, while pure copper is deposited on the copper plate. The molecular arrangements during polarization and after discharge are shown (Fig. 50) in the rows of molecules *a*, *b* respectively.

The result of the action of the cell is that zinc is worn away, zinc sulphate being formed in the acid cell, while the

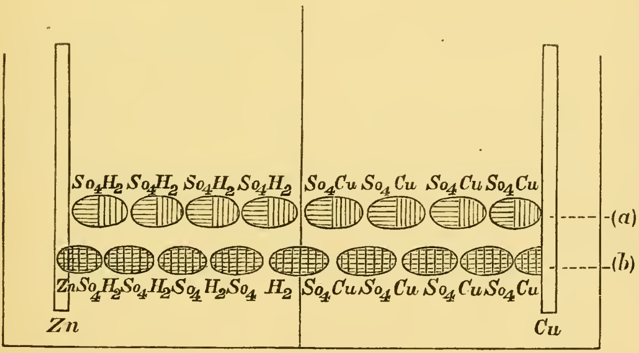

Fig. 50.

copper sulphate is partly replaced by hydrogen sulphate in the salt cell, and copper is deposited on the copper plate or rod.

The electromotive force is represented by

$$Zn \mid H_2SO_4 + H_2SO_4 \mid CuSO_4 + CuSO_4 \mid Cu + Cu \mid Zn.$$

Groves' and Bunsen's cells illustrate the same principle. In them the nitrous fumes given off at the carbon or platinum plate being very soluble in nitric acid do not polarize the plate. These cells have a superior electromotive force to Daniell's cell, but are not so cleanly in working nor so durable.

152. The cells last alluded to are called constant, since the only limit to the working of the cell is apparently the exhaustion of some of the materials which compose it. There is another obstacle called *local action.* It is well known that commercial zinc contains impurities, and also that its density in different parts is very different, while the production of pure and homogeneous zinc would be expensive, if not impossible. The consequence of this want of uniformity is to make the difference of potential between the zinc and fluid different at different parts of their common surface, and galvanic circuits are set up through the zinc itself, which rapidly consume it, and interfere entirely with the action of the cell. To avoid this, the zinc plate used is rubbed over with mercury, which forms a pasty amalgam with the zinc, gives the latter a uniform surface for the action of the acid, and prevents the *local circuits.* The mercury itself is not attacked by the acid, but seems to improve the action of the cell by raising the difference of potential at the zinc plate.

By this means and the employment of various contrivances for ejecting the reduced zinc and supplying the other substances, batteries of the constant class can be kept in working order (as for telegraph purposes) for months without further care than the occasional filling up with acidulated water.

153. The power of a galvanic cell may be increased to an unlimited extent by increasing the number of cells and arranging them in various combinations or batteries: the combination most suitable being determined by the circumstances of each particular case. It will be right here to consider the electromotive force in two arrangements, by compounding which all others are produced. These arrangements are the compound and simple circuit, or as they are now called 'series' and 'parallel.'

In the compound circuit or series all the cells are arranged so that the platinode of one cell is joined to the zincode of the next, the circuit being completed by joining the zincode of the first cell to the platinode of the last.

The arrangement with three cells, A, B, C, can be illustrated thus.

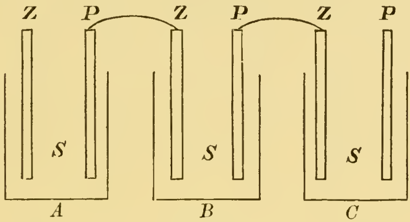

Fig. 51.

If S be the fluid and the zincode of A be to earth, the potential at the zincode of B is

$$Z \mid S+S \mid P+P \mid Z \text{ or } E.$$

The rise of potential between the zincodes of B and C will be again E, making a total rise of $2E$.

Similarly, if there be n cells in series, the rise of potential in all the cells is nE. On this principle batteries have been constructed from which, without joining the terminals, sparks have been produced, Leyden jars charged, attraction and repulsion illustrated, and in fact all the phenomena of statical electricity exhibited. For these purposes several thousand cells must be joined in circuit, and each cell carefully insulated. For an account of these experiments consult Mr Gassiot's Memoir in *Phil. Trans.* for 1844.

154. An illustration of cells in series is seen in Volta's crown of cups and in his pile. The last illustration (Fig. 51) is precisely his crown of cups.

In Volta's pile, a series of zinc and copper plates are arranged in the following order—

Zn S Cu Zn S Cu Zn S Cu......Cu,

S denoting the fluid part of the circuit, which consists of

pieces of flannel soaked in the fluid, generally acidulated water. The contiguous Cu, Zn are soldered together to prevent the fluid soaking in between them. The theory of the pile is precisely the same as that of the compound circuit, the difference of potential of the terminals being simply proportional to the number of metal pairs.

To the same class belong the so-called *dry piles*, the best known of which is Zamboni's, used in the Bohnenberger Electroscope. This pile consists of one to two thousand couples consisting of paper tinned on one side and on the opposite coated with manganese binoxide. In these piles there is an appearance of an electromotive force without a decomposable body. The fact seems to be that some of the elements of the pile (sheets of paper, for instance) are very hygroscopic, and perform the function of the fluid. These piles are found to suspend their action when thoroughly dried and to regain it when left exposed to the damp of the air. In others glass or shellac seems to take the place of the fluid. With them the action of the pile improves on warming, and the reason seems to be that these substances when warm are decomposed by the circuit. That glass belongs to this class of conductors is shown by the fact that if a current is passed for some time through two platinum plates immersed in molten glass, the plates are polarized, and this can only happen, as far as is known, when the substance interposed is decomposed by the current. The glass, however, need not be molten to produce this effect. If a test tube of glass be filled with mercury and dipped in another vessel containing mercury, and if the mercury within and without the tube be connected with a battery enclosing also a galvanometer, no current will pass as long as the glass is cold; but if the mercury be gradually heated at a temperature even below 100° C. a current begins to pass through the glass. If after the current has passed a short time the battery be thrown out of circuit, leaving the galvanometer still in the circuit, a current due to the polarization of the glass will pass in an opposite direction to the battery current, proving that glass even while in a solid state is decomposed by the current.

155. In the arrangement known as simple circuit or parallel all the zincodes are joined to one terminal and all the platinodes to another, the circuit being completed by joining these terminals. The arrangement with these cells will be as follows:

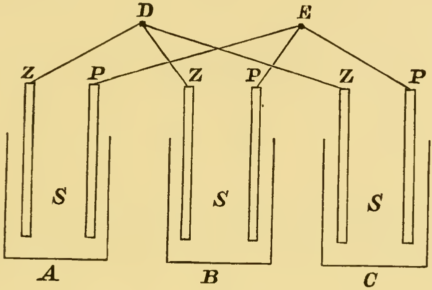

Fig. 52.

all the zincodes being connected with a terminal D, and all the platinodes with a terminal E.

In this case all the zincodes are at the same potential, and all the platinodes at the same potential. The consequence is that the difference of potential between D and E will be only that due to a single cell, or will be simply

$$Z \mid S + S \mid P + P \mid Z.$$

This arrangement is in fact electrically identical with a single cell containing plates of three times the area, which of course in no way affects the electromotive force.

CHAPTER VI.

OHM'S LAW.

156. We have explained above in connection with Faraday's Theory of Induction, the state of a medium acted on by electrical forces (Art. 75). We then established

(i) That in each molecule there was a separation of electricity along the line of force through the molecule, the quantity separated across any equipotential surface being measured by $\pm \frac{F}{4\pi}$ per unit of area, where F is the resultant force at the point.

(ii) That this electrical separation produces or is produced by a strain in the medium along the lines of force, from which strained state the medium tends to return to a neutral state by a discharge from molecule to molecule through the medium.

(iii) That this discharge constitutes conduction resulting in a transfer of the positive electricity separated to the place of lower, and of the negative electricity separated to the place of higher potential, the lines of flow being the lines of force, and the quantity of electricity neutralized along a tube of force being $\pm \frac{1}{4\pi} F\sigma$.

The chief difference between the case there considered and our present problem is that here we have two parts of the conductor kept at constant potentials, so that as soon as one discharge has taken place, the strained state returns

again owing to a new separation of electricities, and we get so rapid a series of discharges, that it cannot be distinguished from a continuous current.

157. We may still assume that the strain at any point in the conductor is measured by $\frac{1}{4\pi}F$, when F is the resultant force at the point, and since good or bad conduction consists only in easy or difficult transmission of electricity, the rate of flow at any point in a given body is proportional to the force at that point. But experiment shows us that different bodies transmit the current in very different degrees, and consequently in different bodies the rate at which the series of charges and discharges succeed each other is different. We shall assume therefore that the rate of flow at any point is measured by the product $c.F$, where c depends on the body, and F measures the force of electrical separation at the point. The quantity c depends on the substance and condition of the body, changing when its temperature or molecular condition varies.

Now to measure the rate of flow of a stream of water we should take a unit of area perpendicular to the stream-lines, and compute the quantity transmitted through it in a certain time. The same method is applied in electricity, and we will suppose c so chosen, that cF measures the quantity transmitted per second across a unit of area of an equipotential surface over which the average force is F. If then a small tube of force be taken whose area is σ, the quantity transmitted per second across any section of the tube will be $cF\sigma$, and since $F\sigma$ is constant throughout the tube, the quantity transmitted per second, at whatever point in the tube it be measured, will be the same.

The quantity c depends entirely on the substance of the conductor, and is called its 'specific conductivity.'

DEF. SPECIFIC CONDUCTIVITY *of a substance may be measured by the quantity of electricity transmitted per second across a unit of area of an equipotential surface, at which the electric force has unit value.*

158. Again, if the conductor be bounded by a tube of force, the quantity transmitted along it per second will be measured by $\Sigma cF\sigma$, and this quantity is called the 'strength of the current' in the tube.

DEF. STRENGTH OF CURRENT *in any tube of force is measured by the quantity of electricity transmitted per second along the tube of force.*

The principle shown above, that the strength of the current in all parts of a tube of force is the same, is often expressed by the phrase 'homogeneity of the circuit.' Since however this strength depends on c the current will not be at once homogeneous unless the substance and temperature of the conductor are the same throughout.

159. Since there is a transfer of the opposite electricities in opposite directions along the tube, it is impossible to speak strictly of the direction of the current, but as most of the phenomena depend on the directions assumed by the opposite currents, it is convenient to define the direction of flow of positive electricity as the direction of the current.

160. Prop. I. To find the strength of current in a conductor on which two surfaces bounded by closed curves are kept at constant potentials.

First, let the conductor be a cylinder whose two ends are at potentials V_1 and V_2.

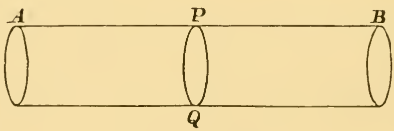

Fig. 53.

Let the end A (Fig. 53) be at V_1 the higher potential, and the end B at V_2: also let l be the length, s the section, and c the specific conductivity of the cylinder.

Let I denote the quantity transmitted per second across any section PQ, or the strength of the current. When

the difference of potential is first established, some of the lines of force will cut the surface of the cylinder, and a flow of electricity will take place along them, producing a superficial distribution, which combined with the constant electrification of the ends, will make the lines of force parallel to the length of the cylinder. The tubes of force will then be cylindrical, the rate of change of potential along them constant, and the flow of electricity will be steady. The rate of change of potential or the electric force along the cylinder will be equal to

$$\frac{V_1 - V_2}{l}.$$

Hence the quantity transmitted per second across any element of the section of the tube whose area is σ will be

$$cF\sigma = c \cdot \frac{V_1 - V_2}{l} \cdot \sigma.$$

Hence the whole quantity transmitted across a complete section will be

$$I = c\,\frac{V_1 - V_2}{l}\,s = \frac{cs}{l}\,(V_1 - V_2).$$

The quantity $\frac{cs}{l}$, which depends only on the substance and dimensions of the cylinder, is called its conductivity, and measures the quantity transmitted per second when the ends are at unit difference of potential.

Secondly. If the conductor be of any form.

The same reasoning may be extended to this case, as the first instantaneous effect of the flow of electricity is to produce a distribution on the surface such that every tube of force shall proceed from one to the other of the given surfaces, after which we have a steady flow of electricity along the tubes of force.

The amount transmitted through any tube is measured, as we have shown, by $cF\sigma$.

If the surfaces be A_1, A_2, and at potentials V_1, V_2, it

follows (from Art. 85) that, neglecting all induction except between A_1 and A_2, the quantity of electricity on these surfaces is $\pm C(V_1 - V_2)$, where C is the capacity of the system A_1, A_2, or $-q_{12}$ according to the notation of that Article. Hence the whole quantity is proportional only to the difference of potential between the surfaces. Again (Art. 69), the law of distribution of electricity over A_1 and A_2 is determinate, and therefore the density at each point is proportional simply to the difference of potential. And the force just outside A_1 is equal to 4π times the density, and is therefore also proportional to $(V_1 - V_2)$; and therefore the value of $F\sigma$ through any tube of force passing from A_1 to A_2 is proportional to $V_1 - V_2$. Let now $F_1\sigma$ be the value of $F\sigma$ when the difference of potential between A_1 and A_2 is unity, and we shall have generally for every such tube of force

$$F\sigma = F_1\sigma (V_1 - V_2).$$

Hence the quantity of electricity transmitted by that tube of force per second

$$= cF_1\sigma (V_1 - V_2).$$

Taking all such tubes of force and adding together the corresponding current strengths we shall have the whole current strength between A_1 and A_2 or

$$\begin{aligned} I &= \Sigma cF_1\sigma (V_1 - V_2) \\ &= c\Sigma F_1\sigma \,.\, (V_1 - V_2). \end{aligned}$$

The coefficient $c\Sigma F_1\sigma$ depends only on the geometry and substance of the conductor and will be defined as its conductivity.

DEF. CONDUCTIVITY OF A CONDUCTOR *is the quantity of electricity which flows per second between two given surfaces on it which are kept at unit difference of potential.*

The numerical value of the conductivity is $c\Sigma F_1\sigma$, where F_1 is the force over the area σ on an equipotential surface, and c the specific conductivity. When the conductor is cylindrical, it is represented by $\frac{cs}{l}$, where s is the area of section, and l the length.

161. In practice we always use the *resistance* instead of the conductivity.

DEF. RESISTANCE *of a given conductor is numerically equal to the reciprocal of its conductivity.*

For any conductor its value is given by

$$\frac{1}{c\Sigma F_1\sigma},$$

and for a cylindrical conductor by

$$\frac{1}{c} \times \frac{\text{length}}{\text{area of section}},$$

where c is the specific conductivity.

The quantity $\frac{1}{c}$ is termed the *specific resistance* of the conductor, and is often denoted by the letter ρ. We notice that the specific resistance is the measure of the resistance of a mass in which the length and cross-section are each unity. Thus the specific resistance of a body is the resistance offered by a c.cm. of the body to a current passing through it parallel to one set of edges. The resistance of any wire or cylindrical body made of this material is given by

$$\rho \times \frac{\text{length in cm.}}{\text{cross section in sq. cm.}}.$$

162. The preceding proposition can be now put in this form: if two parts of a conductor be kept at potentials V_1 and V_2, and if R be the resistance of the conductor, and I the strength of the current,

$$IR = V_1 - V_2.$$

To represent this formula graphically in case of a cylindrical conductor let the abscissa AB (Fig. 54) represent the resistance of the conductor, and at A, B set up ordinates AC and BD representing the potentials at those two points, and join CD.

At a point P in AB, draw an ordinate PQ. Then since $IR = V_1 - V_2$ for any portion of the circuit, it is clear that if V be the potential at P,

$$I = \frac{AC - V}{AP} = \frac{AC - BD}{AB};$$

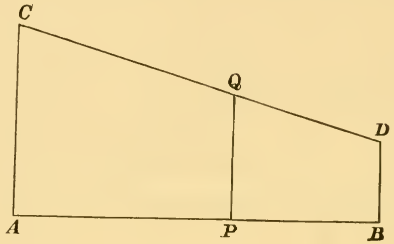

Fig. 54.

hence $V = PQ$, the ordinate drawn to CD, or CD shows the law of fall of potential from A to B referred to resistance.

But when the conductor is cylindrical, the resistance is proportional to the length, and to each point on AB corresponds a point on the conductor dividing its length in the same ratio.

The same figure shows that we may graphically represent the current strength, $\frac{AC - BD}{AB}$, as the cotangent of the angle ACD, or as the tangent of the elevation of the line of potential.

163. Prop. II. In any voltaic circuit if E be the whole electromotive force and R, r the resistances of the conducting wire and fluid in the cell respectively, then the strength of current is given by $I = \frac{E}{R+r}$. Ohm's Law.

Let Z, P, K (Fig. 55) be the zincode, platinode, and junction of the two metals respectively, the abscissæ representing, as in the last Article, the resistances of the parts of the circuit solid and liquid. Owing to the changes in potential at the different contacts, the line of potential will be a broken line.

The law of change we do not at present know, except at the junctions, the potential at Z_1 being the same as at Z,

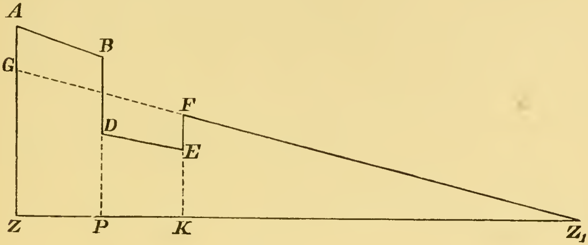

Fig. 55.

the difference of potential at Z being denoted by $Z \mid S$, where S denotes the fluid, that at P by $P \mid S$, and that at K by $P \mid Z$.

Since the conductor throughout is not homogeneous, the circuit will not at first be homogeneous, but there will be a storing up of electricity at the different junctions. This storing up will bring the junctions to such potentials that the current strength between them is uniform. The current then becomes steady.

Let I represent its strength, and let V_1 be the potential in the fluid next the platinode, and V_2 the potential in the platinode wire at its junction with the zincode

$$I = \frac{Z \mid S - V_1}{ZP} = \frac{V_1 - P \mid S - V_2}{PK} = \frac{V_2 + P \mid Z}{KZ_1}$$
$$= \frac{Z \mid S - V_1 + V_1 - P \mid S - V_2 + V_2 + P \mid Z}{ZP + PK + KZ_1}$$
$$= \frac{Z \mid S + S \mid P + P \mid Z}{ZP + PZ_1}.$$

But the numerator represents the electromotive force E, and the denominator the sum of the internal and external resistances. Hence we have

$$I = \frac{E}{R + r},$$

which is known as Ohm's law.

The formula also shows that the line of potential is in a constant direction, and its direction may be found by setting off as abscissa the whole resistance in circuit as ZZ_1, and as ordinate the whole electromotive force as ZG. The line GZ_1 gives us the law of fall of potential in the circuit, omitting of course the discontinuities at the various junctions.

The same reasoning can clearly be applied to any system whatever of conductors arranged in linear series, and with any number of electromotive forces among them. We shall have in all cases if R be one of the resistances, E one of the electromotive forces, and I the current strength,

$$I = \frac{\Sigma E}{\Sigma R}.$$

Although the reasoning by which we have arrived at Ohm's law depends on molecular actions, which are assumed, but cannot be put to experimental test, the law itself has been subjected to the most rigorous experiment, and may be classed in point of certainty with the best ascertained physical laws.

164. In the formula $I = \dfrac{E}{R+r}$, there are three quantities which require to be measured, and it will be convenient here to remind the student of the units in which we have assumed them measured.

(i) *Electromotive force* is difference of potential, and its unit is the unit difference of potential, as defined in Chap. III.

(ii) *Current strength* is the quantity of electricity transmitted per second along a conductor, and its unit the strength of a current in which a unit quantity passes per second.

(iii) *Resistance* is a new idea, and must be measured in accordance with the above formula by the resistance of a conductor which allows a unit of electricity to flow per second through it, the two ends being kept at a unit difference of potential.

These are the units which we have used in theory, but they would be very inconvenient in practice. The practical units depend on electromagnetic phenomena, and we must defer their precise definition till we come to that part of our subject. We will merely state now that

(i) *Electromotive force* is measured by the *volt*, which is about equal to that of an 'open' Daniell cell, i.e. a Daniell cell with the terminals disconnected.

(ii) *Resistance* is measured by the *ohm*, which is the resistance of a certain standard coil of wire at a certain specified temperature. It is found by experiment that the resistance of metals rises with temperature, and it is the low resistance-variation of german silver with temperature which makes it one of the most suitable substances for the construction of Resistance coils. The resistance of glass and carbon among solids, and of liquids generally, falls with increase of temperature.

(iii) *Current strength* is measured by the *ampere;* the ampere being the current in a circuit in which the electromotive force is one volt, and the total resistance one ohm.

The two following are also used:

(iv) *Quantity* is measured by the *coulomb*, which is the quantity of electricity transmitted in one second by a current of one ampere.

(v) *Capacity* is measured by the *farad*, which is the capacity of a condenser charged to one volt by a coulomb of electricity.

We shall assume in future, unless *absolute* units are referred to, that quantities are measured in these terms. For measuring them we require in practice a galvanometer, a set of resistance coils, a cell whose electromotive force is known, and a condenser of known capacity.

165. Prop. III. To find the current strength when n cells each of resistance r and electromotive force E are arranged in parallel.

We have already shown that in this case the electromotive force is unaltered, the arrangement being equivalent to

a single cell in which the plates are n times as large, and since the resistance of a cylindrical conductor varies inversely as area of section, if r be the internal resistance of a single cell that of the battery is only $\frac{r}{n}$. Ohm's formula therefore becomes

$$I = \frac{E}{\frac{r}{n} + R} = \frac{nE}{r + nR}.$$

COR. When the internal resistance is small compared to the external, this formula is equivalent to $I = \frac{E}{R}$, and the current strength is not increased by increasing the number of cells. If, however, R be small compared to r, or the external resistance be very small, the formula is equivalent to $I = \frac{nE}{r}$, or the current strength is increased in proportion to the number of cells.

166. Prop. IV. To find the current strength when n cells are arranged in series.

Here we have shown that the whole electromotive force is nE. Each cell, however, introduces a fresh resistance, and the whole resistance in the battery becomes nr.

Hence Ohm's formula gives

$$I = \frac{nE}{nr + R}.$$

COR. If r be small compared to R, or when the internal resistance is small, $I = \frac{nE}{R}$, or the current strength is increased n-fold. But if nr be large compared to R, the formula reduces to $\frac{nE}{nr} = \frac{E}{r}$, or the current is not increased by increasing the number of cells.

167. Remembering the construction for the line of potential, we can illustrate graphically the two last propositions.

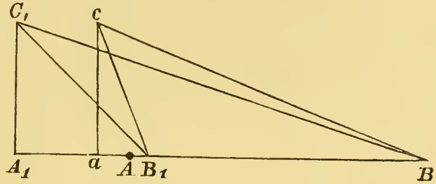

Fig. 56.

Let AA_1 be the internal resistance, and A_1C_1 the electromotive force of a single cell. If there be n cells in parallel, the internal resistance is $Aa = \frac{1}{n} AA_1$, and the electromotive force is $ac = A_1C_1$.

Then if the external resistance be small compared to AA_1 (as AB_1) the current strength is increased in ratio $\tan aB_1c$ to $\tan A_1B_1C_1$.

But if the external resistance be several multiples of AA_1 as AB, the increase is only in the ratio $\tan aBc$ to $\tan A_1BC_1$, nearly an equality, or the battery gives scarcely more than a single cell.

168. In the case of cells in series, the resistances of the successive cells are represented by Aa_1, a_1a_2, $a_2a_3 \ldots a_{n-1}a_n$,

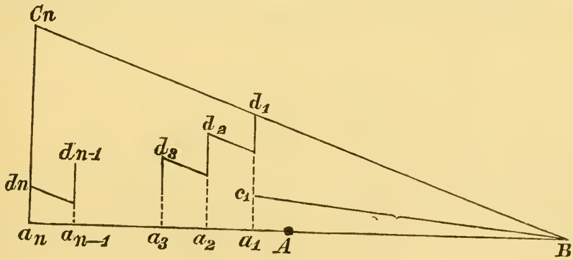

Fig. 57.

and the electromotive force is $a_nC_n = n \,.\, a_1c_1$, or n times the electromotive force of a single cell. If AB be the external

resistance, the line of potential will *outside the battery* be given by BC_n. For the law of change in the battery we clearly have a broken line discontinuous at each junction, but if we assume the whole rise in each cell to take place at the zinc plate, assuming the last zinc in connection with B, the line will be represented by the broken line $d_1 d_2 \ldots \ldots d_n$. If $a_1 c_1$ represent the electromotive force of a single cell, the increase in strength of current will be in the ratio tan ABC_n to tan ABc_1. This is large when AB is considerable.

If AB be very small so that A and B coincide, the line of potential will be as in the following figure, in which,

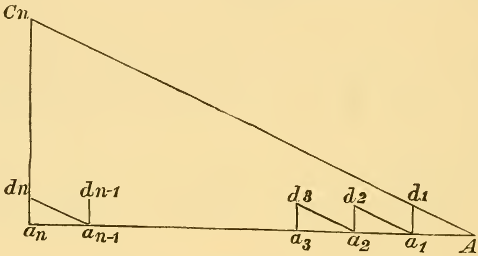

Fig. 58.

from the construction of the figure, it is clear that the fall in each cell of the battery is equal to the rise at each zinc plate. There is in this case no gain from using a compound circuit.

169. The corollaries to the two last propositions show that when the external resistance is very large there is no

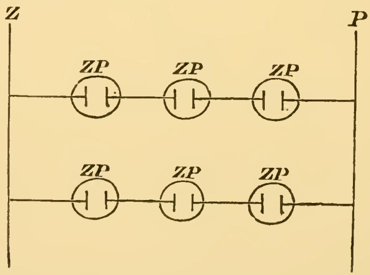

Fig. 59.

advantage obtained by arranging the cells in parallel, and when very small there is no advantage in arranging them in series. When the resistance is moderate we obtain a greater current than by either arrangement, by a combination of the two, which may be called *mixed circuit*, and is illustrated in the six cells of Fig. 59, in which the vertical rows are in parallel, and the horizontal rows in series; the arrangement being the same as that of three cells arranged in series, each cell having plates twice as large as those of the single cell.

170. Prop. V. To find the current-strength due to pq cells arranged in q horizontal rows of p cells, the cells in each row being in compound circuit and the successive rows in simple circuit.

Here the electromotive force is clearly pE, and the resistance in the battery $p \times \frac{r}{q}$, since the arrangement is the same as that of p cells whose plates are q times as large as the plates of each cell.

Hence Ohm's formula gives for the current strength

$$I = \frac{pE}{\frac{p}{q}r + R}.$$

171. Prop. VI. To find the best arrangement of n cells when the external resistance is given.

Let them be arranged in q rows of p cells each.

Then we have $$n = pq,$$

and $$I = \frac{pE}{\frac{p}{q}r + R} = \frac{pE}{R + \frac{r}{n}.p^2}.$$

We require the value of p which makes the last expression a maximum. This is easily seen to be when the two terms in the denominator are equal. For it is of the algebraical form

$$\frac{ap}{b + cp^2} = \frac{a}{\frac{b}{p} + cp}.$$

Now $$\frac{b}{p} + cp = 2\sqrt{bc} + \left(\sqrt{\frac{b}{p}} - \sqrt{cp}\right)^2,$$

and a perfect square is necessarily positive. Hence the right-hand side has the smallest possible value when the square vanishes; that is when

$$\frac{b}{p} = cp \text{ or } b = cp^2.$$

In the particular case above I is a maximum when

$$R = \frac{r}{n}p^2 = \frac{rp}{q},$$

i.e. when the internal and external resistances are equal.

The greatest value of $\frac{pr}{q}$ is clearly when $p = n$ and $q = 1$; then $R = nr$. If R has this or any greater value, the series is the best. If R is less than this we must choose p and q so as to satisfy as nearly as possible the condition of making the external and internal resistances equal.

172. The proposition of the preceding Article may also be treated graphically.

If there be n cells of electromotive force E and resistance R, arranged in q rows of p cells, we have for ϵ the electromotive force and ρ the internal resistance of the battery, the equations

$$\epsilon = pE,$$

$$\rho = \frac{pR}{q},$$

$$n = pq.$$

Hence $$\rho = \frac{p^2R}{n} = \frac{R}{n}\left(\frac{\epsilon}{E}\right)^2;$$

$$\therefore \epsilon^2 = \frac{nE^2}{R} \cdot \rho;$$

an equation which shows that if ρ be an abscissa and ϵ

the corresponding ordinate, the locus of its extremity is a parabola whose latus rectum is $\frac{nE^2}{R}$. Hence in all arrangements of the battery the relation between its internal resistance and electromotive force is represented graphically by the abscissa and ordinate of a parabola.

The only part of the curve practically available will be that between the abscissa $\frac{R}{n}$, when the cells are in parallel and nR when they are in series.

Tracing the curve we shall have the portion between B and C, for instance, available.

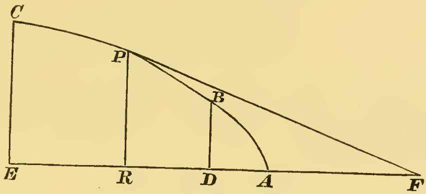

Fig. 60.

If the external resistance be set off to the right of A along the axis, equal suppose to AF, the strength of the current when simple circuited is tan AFB, and when compound circuited tan AFC (Art. 162).

The greatest possible strength of current will be that corresponding to a tangent drawn from F to the parabola, suppose FP; then AR is the internal resistance, PR the electromotive force, and the current strength is given by tan RFP.

By a well-known property of the parabola we have $AR = AF$, or the external and internal resistances are equal.

173. Prop. VII. To investigate the strength of the current and the whole resistance in any divided circuit.

Suppose the potentials at two points A, B to be V and V'. Let the resistances in the various branches ACB, ADB, AEB, &c. be R_1, R_2, R_3... and the current strengths I_1, I_2, I_3...

Then in the respective branches by Prop. I.

$$V - V' = I_1R_1 : V - V' = I_2R_2 : V - V' = I_3R_3 \text{ \&c.}$$

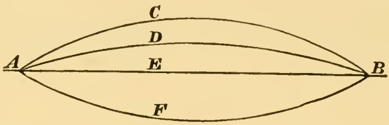

Fig. 61.

Hence $$V - V' = \frac{I_1}{\frac{1}{R_1}} = \frac{I_2}{\frac{1}{R_2}} = \frac{I_3}{\frac{1}{R_3}} = \ldots\ldots$$

$$= \frac{I_1 + I_2 + I_3 + \ldots\ldots}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots}.$$

But the whole current passing is clearly the sum of that passing in each branch.

Hence $$I = I_1 + I_2 + I_3 + \ldots\ldots$$

Let also $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots$$

Hence $$V - V' = \frac{I}{\frac{1}{R}} = IR.$$

But by Art. 162, when

$$I = \frac{V - V'}{R},$$

R is by definition the resistance of the conductor.

Hence for the resistance of any divided circuit we have, if R be the whole resistance and R_1, R_2... the branch resistances,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots$$

and the current in each branch is inversely proportional to its resistance.

174. Prop. VIII. To investigate the current in each branch of any net-work of linear conductors. (Kirchhoff's Laws.)

Any net-work may be resolved into a system of linear conductors, and a system of junctions at which three or more linear conductors meet.

We must begin by giving arbitrary values to the potential of each junction, except two, at which we must suppose the potentials given by connection with a battery or otherwise.

For each linear conductor at whose extremities the potentials are V_r and V_p, whose resistance is R_s, and in which the current is I_s, we have

$$V_r - V_p = I_s R_s \text{ .... } (A),$$

and similarly for each linear portion.

For each junction we know that the same amount of electricity which flows to it must flow from it. Hence if $I_1, I_2, I_3 \ldots$ be the current strength flowing *all to or all from* a given junction, we have the *algebraical equation*

$$I_1 + I_2 + I_3 + \ldots\ldots = 0,$$

or

$$\Sigma I = 0 \text{ .... } (B).$$

The systems of equations (A), (B) always give us enough simple simultaneous equations to find the current in each branch and the potential at each junction.

175. The equations (A) can usually be simplified by regarding the net-work as a set of closed circuits.

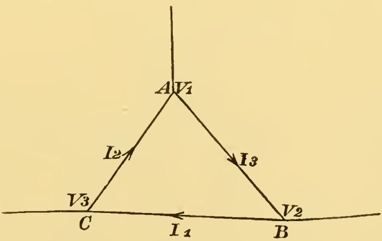

Fig. 62.

In any closed circuit ABC, where there is no impressed electromotive force, we have

$$V_3 - V_1 = I_2R_2,$$
$$V_1 - V_2 = I_3R_3,$$
$$V_2 - V_3 = I_1R_1.$$

Hence $$I_1R_1 + I_2R_2 + I_3R_3 = 0,$$

or $$\Sigma IR = 0.$$

If there be in any branches of the circuit impressed electromotive forces E, E_1, E_2, E_3......, we have similarly

$$\Sigma IR = \Sigma E \text{ ....} (C).$$

The sets of equations (B), (C) will be generally sufficient to determine the current in each branch.

176. Prop. IX. To investigate the current-strength in a system consisting of six conductors joining four points (a quadrilateral and its diagonals), four of the branches having in them electromotive forces.

Let $ABCD$ represent such a system. Let the current-strengths in the branches be I_1, I_2......I_6, the electromotive forces E_1...... E_4, and the total resistances R_1...... R_6, as in figure.

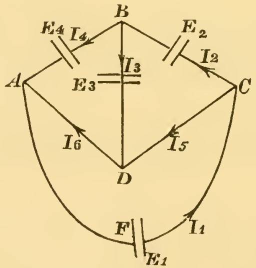

Fig. 63.

The equations for current-strengths are—

$$\left.\begin{array}{ll} \text{from } ABCD; & E_2 + E_4 = I_2R_2 + I_4R_4 - I_5R_5 - I_6R_6 \text{...(i)} \\ \text{from } ABD; & E_3 - E_4 = I_3R_3 - I_4R_4 + I_6R_6 \text{ ....(ii)} \\ \text{from } ADCF; & E_1 = I_1R_1 + I_5R_5 + I_6R_6 \text{ ....(iii)} \end{array}\right\} (C);$$

$$\left.\begin{array}{lll} \text{at } A; & 0 = I_1 - I_4 - I_6 & \dots\dots\dots\dots\dots\dots\text{(iv)} \\ \text{at } B; & 0 = I_2 - I_3 - I_4 & \dots\dots\dots\dots\dots\dots\text{(v)} \\ \text{at } C; & 0 = I_1 - I_2 - I_5 & \dots\dots\dots\dots\dots\dots\text{(vi)} \end{array}\right\} (B).$$

The six equations (B), (C) are independent and sufficient to determine the current-strength in each of the six branches.

An important case arises when the current in one branch is independent of the electromotive force and resistance in another branch, in which case these two branches are said to be *conjugate* to each other.

177. Prop. X. To show that in the system of six conductors joining four points the diagonals are conjugate to each other if the products of the resistances in the opposite sides be equal.

We must proceed to solve the six equations of the last proposition to find I_3. We shall adopt the method of indeterminate multipliers, multiplying the equations (ii)—(vi) by λ_1, $\lambda_2 \dots \lambda_5$ respectively. If we add the resulting equations together and equate separately to zero the coefficients of I_1, I_2, I_4, I_5, I_6, we shall have five equations to determine $\lambda_1 \dots \lambda_5$, and the remaining equation for I_3—

$$I_3(\lambda_1 R_3 - \lambda_4) = E_2 + E_4 + \lambda_1(E_3 - E_4) + \lambda_2 E_1 \dots\dots\text{(vii)}.$$

I_3 will by this equation be independent of E_1 if $\lambda_2 = 0$.

Writing down the five equations for λ's, with the extra condition $\lambda_2 = 0$, we have

$$\left.\begin{array}{r} \lambda_3 + \lambda_5 = 0 \\ R_2 + \lambda_4 - \lambda_5 = 0 \\ R_4 - \lambda_1 R_4 - \lambda_3 - \lambda_4 = 0 \\ -R_5 - \lambda_5 = 0 \\ -R_6 + \lambda_1 R_6 - \lambda_3 = 0 \end{array}\right\} \dots\dots\dots\dots\dots\text{(viii)}.$$

The condition that these equations can be satisfied simultaneously is found by eliminating λ_1, λ_3, λ_4, λ_5 from them. The result is easily seen to be

$$R_4 R_5 - R_2 R_6 = 0 \dots\dots\dots\dots\dots\dots\dots\text{(ix)}.$$

If (ix) is satisfied, (viii) can be satisfied, and (vii) will then be satisfied with the extra condition $\lambda_2 = 0$. In this

case I_3 is independent of E_1, and since R_1 enters none of equations (vii), (viii), I_3 must be also independent of R_1.

Cor. 1. Conversely the current in AC will, if (ix) be satisfied, be independent of the electromotive force and resistance in BD.

Cor. 2. If $I_3 = 0$, which will happen if (ix) be satisfied and E_1 be the only electromotive force, it is clear that R_3 enters none of the equations (B), (C), and the currents in all the branches will be unaltered by making or breaking contact in BD.

178. Prop. XI. To find the time of discharge of a given electrified system.

Let two surfaces A, B be at potentials V_1 and V_2, and let R be the resistance of the medium interposed between them, all measured in absolute units. Let also C be the electrostatic capacity of the system; then the quantities of electricity $\pm C(V_1 - V_2)$ tend to neutralize each other by conduction through the medium.

Let us assume that v is the difference of potential, and $\pm q$ the quantities of electricity after a time t, and that v' and $\pm q'$ represent the same things after $t+\tau$, where τ is a very short interval.

By Ohm's law $I = \frac{v}{R}$, where I is quantity of flow per second.

Hence the quantity which flows through in time τ

$$= I\tau = \frac{v\tau}{R}.$$

Hence $$q - q' = \frac{v\tau}{R}.$$

But $$q - q' = C(v - v');$$

$$\therefore CR(v - v') = v\tau.$$

Hence $$\tau = CR\frac{v - v'}{v}.$$

By Art. 37, when $\frac{v-v'}{v}$ is very small, as will be the case here, we may put

$$\frac{v-v'}{v} = -\log\left(1 - \frac{v-v'}{v}\right)$$

$$= -\log\frac{v'}{v};$$

$$\therefore \tau = +CR\log\frac{v}{v'},$$

$$= CR(\log v - \log v').$$

The same proposition will hold for any number of very short intervals; we shall have, if v be the difference of potential after a time t from charging,

$$t = CR\{\log(V_1 - V_2) - \log v\};$$

$$\therefore \frac{t}{CR} = \log\frac{V_1 - V_2}{v};$$

$$\therefore v = (V_1 - V_2)\,\epsilon^{-\frac{t}{CR}},$$

which gives the potential at any time t.

If we observe in what time $v = \frac{1}{n}(V_1 - V_2)$, or the difference of potential falls to one n^{th} of its first value,

$$\frac{1}{n} = \epsilon^{-\frac{t}{CR}}, \text{ or } t = CR\log n.$$

179. Prop. XII. If K be the specific inductive capacity, and ρ the specific resistance of a substance, and if C be the electrostatic capacity of any condenser made of that substance, and R its resistance to the passage of electricity; to prove that $\mathrm{CR} = \frac{1}{4\pi}.\rho\mathrm{K}$ in absolute measure.

It is assumed that lines of force proceed exclusively from one surface to the other of the body under consideration.

To find the electrostatic capacity, we notice that along any tube of force $F\sigma$ is constant, and at either bounding

surface $F\sigma = 4\pi\rho\sigma = 4\pi q$. Hence the whole charge Q on either surface is given by

$$4\pi Q = \Sigma F\sigma,$$

and if F_1 be computed on the assumption that the opposite surfaces differ in potential by one unit (Art. 160),

$$Q = C \text{ when } F = F_1;$$

$$\therefore\ 4\pi C = \Sigma F_1\sigma,$$

if the dielectric be air. In the supposed case where the dielectric has specific inductive capacity K,

$$C = \frac{K}{4\pi}\,\Sigma F_1\sigma.$$

But it has already been shown (Art. 161)

$$R = \frac{1}{c\Sigma F_1\sigma} = \frac{\rho}{\Sigma F_1\sigma},$$

where $\frac{1}{\rho} = c$, the specific conductivity;

$$\therefore\ C = \frac{K}{4\pi}\cdot\frac{\rho}{R},$$

$$\text{or } RC = \frac{1}{4\pi}\rho K.$$

COR. We have shown in the last Article that if t be the time of falling to $\left(\frac{1}{n}\right)^{\text{th}}$ of charge,

$$RC = \frac{t}{\log_\epsilon n}.$$

Hence we have

$$RC = \frac{1}{4\pi}\rho K = \frac{t}{\log_\epsilon n}.$$

180. Prop. XIII. To calculate the amount of heat developed in any portion of a galvanic circuit.

Let the potentials in absolute measure at the extremities of the circuit be V_1 and V_2, R the resistance of the interposed circuit, and I the strength of the current. By defi-

nition of current-strength I units of electricity pass from potential V_1 to potential V_2 per second, and when no external work is done this amount of energy must be converted into heat in the circuit.

Hence the mechanical equivalent of the heat given out per second is $I(V_1 - V_2)$.

Again, if J represent Joule's mechanical equivalent of heat, or the number of ergs imparted to a gramme of water which is warmed from 0° C. to 1° C., and if H be the number of units of heat given out per second, then JH will also represent the mechanical equivalent of the heat given out per second, and we have

$$JH = I(V_1 - V_2).$$

But $$V_1 - V_2 = IR.$$

Hence $$JH = I^2R = \frac{(V_1 - V_2)^2}{R}.$$

This equation is true when J, $V_1 - V_2$ and R are all expressed in absolute measure.

The value of J is $4{\cdot}2 \times 10^7$ and if V and R be measured in volts and ohms we must multiply the right-hand side by 10^7 (Art. 257).

Hence expressing electrical magnitudes in the ordinary units of practice, volts, ohms and amperes, we have

$$H = \frac{I(V_1 - V_2)}{4{\cdot}2} = \frac{I^2R}{4{\cdot}2} = \frac{(V_1 - V_2)^2}{4{\cdot}2 \, . \, R}.$$

H will now give the heat in gram-degrees-C.

Cor. 1. If the conductor be a wire whose specific heat is c, w its weight in grammes, and θ the elevation of temperature per second, then $H = cw\theta$;

$$\therefore \; 4{\cdot}2cw\theta = I^2R = \frac{(V_1 - V_2)^2}{R} = I(V_1 - V_2),$$

a formula giving the elevation in temperature per second owing to the passage of the current, supposing no loss of heat.

Cor. 2. It appears from the last formula that the elevation in temperature is independent of the length of the wire, provided the strength of the current be constant, for θ will be proportional to $\frac{R}{w}$, which is a constant, since both numerator and denominator are proportional to the length.

Cor. 3. If the section of the wire vary, the current remaining of the same strength, it is clear that R varies inversely as the area of section, and w varies directly as the area of section. Hence the quotient $\frac{R}{w}$ will vary inversely as the square of the section, or if the section be similar throughout, inversely as the fourth power of the diameter.

180 *a*. We may use the expression $I(V_1 - V_2)$ to represent the output of energy whenever a current I runs between potentials V_1 and V_2.

If I and V are both in absolute measure this will amount to $I(V_1 - V_2)$ ergs per sec., or if I, V be in amperes and volts it becomes $I(V_1 - V_2)\,10^7$ ergs per sec.

In practice the Watt expresses the rate of working at which 10^7 ergs per second are given out: this therefore represents the output of one ampere running through one volt, the usual definition of the Watt.

Def. *The Watt expresses the rate of output of energy of an electric current in which* 1 *ampere runs through an* E.M.F. *of one volt.* This is of course 10^7 ergs per second, and by Art. 20 is $\frac{1}{746}$ of a horse-power.

Def. *The Joule is the amount of energy developed by one Watt in one second.* This of course equals 10^7 ergs.

181. Prop. XIV. To show that in any divided circuit in which there is a distribution of the current-strength in each branch inversely as its resistance, there will be less heat given out than if the same total currents were distributed in any other way. Principle of Least Heat.

First, let the circuit contain only two branches. Then if I_1, R_1 and I_2, R_2 be corresponding quantities for the two branches, and I the whole current,

$$I = I_1 + I_2,$$

and
$$JH = I_1^2 R_1 + I_2^2 R_2;$$

$$\begin{aligned}\therefore\ JH(R_1 + R_2) &= I_1^2 R_1^2 + I_2^2 R_2^2 + (I_1^2 + I_2^2) R_1 R_2 \\ &= (I_1 R_1 - I_2 R_2)^2 + (I_1 + I_2)^2 R_1 R_2 \\ &= (I_1 R_1 - I_2 R_2)^2 + I^2 R_1 R_2.\end{aligned}$$

The right-hand side will have the least possible value when the first term vanishes, or when

$$I_1 R_1 = I_2 R_2,$$

i.e. the current in each branch is inversely as the resistance.

Hence also H or the heat given out will have its least possible value.

Secondly, in any divided circuit we see that for any two of the branches this relation must hold, or we could redistribute the current in these two so as to evolve less heat without disturbing the current in the other branches. Hence we infer, however many branches there be, there will be least heat evolved when the current in each varies inversely as the resistance, or when the currents are distributed according to Ohm's law.

CHAPTER VII.

PROBLEMS IN VOLTAIC ELECTRICITY.

182. In the following problems we shall endeavour to illustrate the propositions of the preceding Chapter by laying before the student a number of results mostly of the highest importance to the practical electrician.

The chief instruments we shall assume used will be a galvanometer, a box of resistance coils, and a quadrant or other form of electrometer capable of giving absolute measure. The theory of the galvanometer we do not enter into here as it belongs to Magnetism. We shall assume however that the form used is that known as the tangent galvanometer (unless the contrary be stated), in which the strength of the current is proportional to the tangent of the deflection.

It is not generally necessary to determine a current in absolute measure, our problems nearly always depending on the comparison of two currents with the same galvanometer.

183. Prop. I. To investigate the electrical conditions of Wheatstone's Bridge.

Wheatstone's Bridge is only a particular case of the system of conductors investigated in Arts. 176, 177.

This instrument consists essentially of a double divided circuit, two points in the divided branches being joined by a conducting wire. These divided circuits are ABC and ADC, and BD is the joining wire. In the portions AB, BC, CD, DA, are introduced resistances, which we shall call p, q, s, r, and in BD is a galvanometer whose resistance we call g. The current from a galvanic cell, E, enters at A and leaves at C.

We shall denote the electromotive force of the cell by E and its resistance by R.

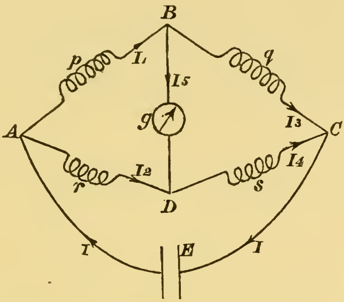

Fig. 64.

The current-strengths in the various branches we denote by I, I_1, I_2, I_3, I_4, I_5, as shown in figure.

Then in circuit $EABCE$ we have

$$E = RI + pI_1 + qI_3 \quad \text{.................(i)},$$

in ABD, $$0 = pI_1 + gI_5 - rI_2 \quad \text{.................(ii)},$$

in BCD, $$0 = qI_3 - sI_4 - gI_5 \quad \text{.................(iii)},$$

at A, $$I = I_1 + I_2 \quad \text{.........................(iv)},$$

at C, $$I = I_3 + I_4 \quad \text{.........................(v)},$$

at B, $$I_1 = I_5 + I_3 \quad \text{.........................(vi)}.$$

These six equations will be found independent, and can be easily solved, giving the strength of the current in each of the six branches.

By (vi), $$I_3 = I_1 - I_5.$$

By (v) and (iv), $I_4 = I - I_3 = I - I_1 + I_5 = I_2 + I_5$.

Substitute in (iii),

$$0 = q(I_1 - I_5) - s(I_2 + I_5) - gI_5,$$

or $$qI_1 - sI_2 - (q + s + g) I_5 = 0.$$

By (ii), $$pI_1 - rI_2 + gI_5 = 0.$$

$$\text{Hence}\quad \frac{I_1}{r(q+s+g)+sg}=\frac{I_2}{p(q+s+g)+qg}=\frac{I_5}{qr-ps},$$

$$\therefore\ =\frac{I_1-I_5}{s(p+r+g)+rg}=\frac{I_3}{s(p+r+g)+rg}\ \text{by (vi)},$$

$$\text{also}\quad =\frac{I_2+I_5}{p(q+g)+q(r+g)}=\frac{I_4}{p(q+g)+q(r+g)}\ \text{by (v)},$$

and

$$=\frac{I_1+I_2}{(r+p)(q+s)+g(p+q+r+s)}=\frac{I}{(r+p+g)(q+s+g)-g^2}.$$

By help of (i) each of the above $=\dfrac{E}{D}$,

where
$$\begin{aligned}D&=R\{(r+p)(q+s)+g(p+r+q+s)\}\\&\quad+pr(q+s+g)+psg\\&\quad+qs(p+r+g)+qrg\\&=R(p+r)(q+s)+pr(q+s)+qs(p+r)\\&\quad+\{R(p+q+r+s)+(p+q)(r+s)\}g.\end{aligned}$$

The currents in all the branches can now be written down.

184. In the use of the bridge for determining an unknown resistance it is introduced in place of p or q and a balance secured by altering the resistances in the other branches until the current in the galvanometer (I_5) vanishes. When this condition is satisfied $qr-ps=0$ and if q be the unknown, and p, r, s known resistances, $q=\dfrac{ps}{r}$.

185. We might at once see by the graphical method that if this relation hold, there is no current in the galvanometer.

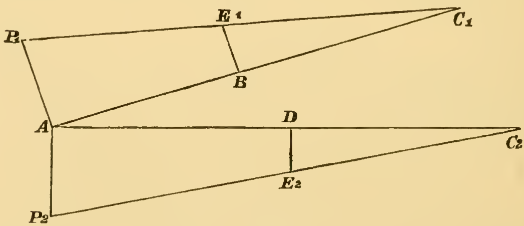

Fig. 65.

Let the resistances ABC and ADC be represented by ABC_1, ADC_2.

Let the differences of potential between the extremities of the conductor be represented by AP_1 perpendicular to AC_1, and AP_2 equal to it perpendicular to AC_2. The line P_1C_1 represents the fall of potential along AC_1, and P_2C_2 along AC_2.

By similar triangles

$$BE_1 : AP_1 :: C_1B : C_1A,$$

and $$DE_2 : AP_2 :: C_2D : C_2A.$$

Hence $$BE_1 = DE_2,$$

if $$C_1B : C_1A :: C_2D : C_2A,$$

or $$C_1B : BA :: C_2D : DA,$$

or $$C_1B \,.\, DA = BA \,.\, C_2D,$$

the relation already found. If B and D be now joined there will be no current in BD, since the extremities are at the same potential.

Cor. It follows that the currents in the other branches will remain unaltered whether the branch BD be open or closed.

186. Prop. II. To find the resistance of a galvanometer coil.

This resistance can be measured by Wheatstone's Bridge just as that of any other conductor. After the magnet has been mounted in its place, the following method, due to Sir W. Thomson, is found to lead to more accurate results.

Place the galvanometer in the branch BC (Fig. 64), and in BD place a contact-breaker instead of a galvanometer. It appears by the corollary to the last Proposition that if the relation $ps = rq$ be satisfied, the galvanometer deflection will remain the same whether contact in BD be made or broken.

We have therefore only to adjust the other resistances until the galvanometer reading does not alter on making or breaking contact in BD, and then the resistance of the galvanometer is given by

$$q = \frac{sp}{r}.$$

187. Prop. III. To measure the internal resistance of a Battery.

1*st Method.* If we make circuit in the battery by pieces of stout wire connecting its poles with the galvanometer, the only external resistance will be that of the galvanometer, g. Then if δ_1 be the observed deflection, and x the unknown internal resistance,

$$c \tan \delta_1 = \frac{E}{x+g},$$

where c depends on the galvanometer.

Introduce now between one pole and the galvanometer a measured resistance r. If δ_2 be the new deflection

$$c \tan \delta_2 = \frac{E}{x+r+g},$$

dividing
$$\frac{\tan \delta_1}{\tan \delta_2} = \frac{x+r+g}{x+g},$$

a simple equation for x.

This method is open to many objections, as the observations are taken with two different current-strengths. From these objections Mance's method seems free.

188. 2*nd Method* (Mance's). In this method is employed a modification of Wheatstone's Bridge, similar to that used

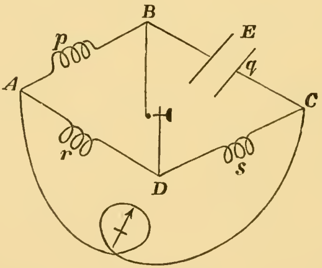

Fig. 66.

by Sir William Thomson for measuring the galvanometer resistance. The cell whose resistance is to be measured is placed in BC, the galvanometer in AC, and a contact-breaker in BD. The arrangement will then be as in the figure.

Now if the resistance in the branches satisfy the condition $ps = qr$, the branches BD and AC are conjugate, and consequently making or breaking contact in BD will produce no effect on the galvanometer in AC. If therefore one resistance be adjustable, we adjust it until the galvanometer is uninfluenced by making and breaking, and we have then for q, the unknown resistance,

$$q = \frac{ps}{r}.$$

189. Prop. IV. To compare the electromotive force of two cells.

1*st Method.* Take one cell and introduce resistance till the galvanometer stands at a certain deflection δ_1. Let r_1 be the resistance introduced; g, R those of the galvanometer and cell; then

$$c \tan \delta_1 = \frac{E_1}{r_1 + g + R}.$$

Add resistance r_1', so that the deflection comes down to δ_2,

$$\therefore \; c \tan \delta_2 = \frac{E_1}{r_1' + r_1 + g + R};$$

$$\therefore \; \frac{1}{c}(\cot \delta_2 - \cot \delta_1) = \frac{r_1'}{E_1}.$$

Next, by introducing resistance into the circuit of the other cell bring its deflection to δ_1. Add resistance (suppose r_2') till its deflection is δ_2 as before.

Then $$\frac{1}{c}(\cot \delta_2 - \cot \delta_1) = \frac{r_2'}{E_2};$$

$$\therefore \frac{r_1'}{E_1} = \frac{r_2'}{E_2};$$

$$\therefore \; E_1 : E_2 :: r_1' : r_2',$$

or the electromotive forces are proportional to the resistances which must be introduced to bring the galvanometer from one fixed reading δ_1 to another δ_2.

The objection to this simple method is that the electromotive forces are subject to variation from a variety of causes when the battery is in action, and the comparison should always be made when no current is passing.

190. *2nd Method. By Clark's Potentiometer.* For this method we require a battery of very constant electromotive force, and a length of fine wire coiled along an ebonite cylinder similar to Wheatstone's rheostat.

Let A be the constant battery, BC the cylinder, and E_1, E_2 the cells to be compared.

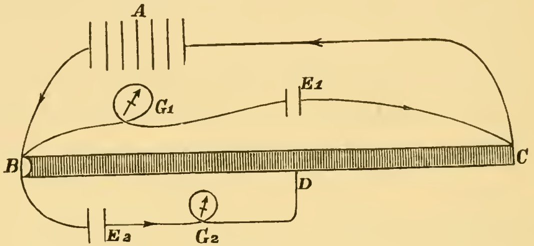

Fig. 67.

Connect the battery A and a variable resistance in the circuit ABC, and make a branch circuit BG_1E_1C containing a galvanometer G_1 and the cell E_1 (which is supposed to have greater electromotive force than E_2), so placed that its current in BC is opposite in direction to that of A. Vary the resistance in AC till the galvanometer G_1 is at zero, when the difference of potential between B and C will equal E_1. Introduce now the second cell E_2 having its negative pole at B, with a second galvanometer G_2. With the positive pole D make contact at successive places along BC till there is no current in G_2. We then have $E_2 : E_1 :: BD : BC$. Hence if a divided scale be attached to BC and graduated from 0 to 100, we can at once read off the electromotive force of E_2 in terms of E_1.

191. Prop. V. To find the position of a "fault."

The term fault is applied in practical telegraphy to any sudden change in the resistance of the line. Faults arise from a variety of accidents to the line-wire such as breakage,

failure of insulation, contact with foreign bodies, &c. and a large number of methods have been proposed for the determination of their position in the line. The following are a few of the most simple:

1*st Method.* For a land-line in which the wire is completely broken, the broken end not making earth.

In this case the resistance will be the insulation resistance up to the fault, the only escape of electricity being by the insulating supports, or through the gutta-percha sheath which surrounds the wire. We must compare it therefore with the insulation resistance (supposed known) of the unbroken line, and assuming the line uniform, it is clear that the resistance is inversely proportional to the length of cable tested, and we shall have the proportion,

$$\text{Resistance of faulty line} : \left\{\begin{matrix}\text{Resistance of complete line with} \\ \text{distant end insulated}\end{matrix}\right.$$

$$:: \text{length of line} : \text{distance of fault},$$

whence the distance of the fault is found.

192. 2*nd Method.* When the wire is severed and the broken end makes complete earth.

Here the resistance will be diminished, since the electricity escapes to earth at the fault, instead of at the further end. Hence the resistance will be directly proportional to the length, and we have

$$\left.\begin{matrix}\text{Resistance of whole line with} \\ \text{end to earth}\end{matrix}\right\} : \text{Resistance of faulty line}$$

$$:: \text{length of line} : \text{distance of fault},$$

whence again the position of the fault can be found.

193. 3*rd Method.* When the wire is not completely broken, but makes partial earth. *Blavier's Formula.*

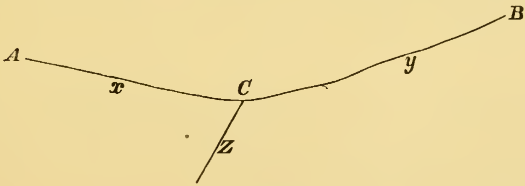

Fig. 68.

Let R be the resistance of the line AB when perfect, S the resistance of the faulty line measured from A when B is to earth, T the resistance of the faulty line when B is insulated.

If x, y, z be the resistances of the portions AC, CB, and of the fault at C, we have

$$R = x + y \text{ ....(i)},$$

$$S = x + \frac{yz}{y+z} \text{ ....(ii)},$$

$$T = x + z \text{ ....(iii)},$$

three equations for x, y, z.

By (i), $\qquad y = R - x,$

and by (iii), $\qquad z = T - x.$

Substituting in (ii),

$$S = x + \frac{(R-x)(T-x)}{R+T-2x},$$

$$(R+T)S - 2xS = (R+T)x - 2x^2 + RT - (R+T)x + x^2;$$

$$\therefore x^2 - 2xS = RT - (R+T)S;$$

$$\therefore (x-S)^2 = RT - (R+T)S + S^2$$

$$= (R-S)(T-S);$$

$$\therefore x = S - \sqrt{(R-S)(T-S)},$$

which is Blavier's formula.

194. These methods all have the imperfection of assuming that the resistance at the fault remains constant (or vanishes) during the measurements, and of neglecting the leakage through the insulating sheath or supports. The following, due to Dr Siemens, seems free from this objection.

4th Method. Siemens' Method for a submarine cable or land-line.

Let AB represent the faultless cable insulated at both ends, AC, BE equal but variable resistances, CD, EF constant equal resistances. At D the positive pole of a battery is attached, and at E the negative pole of an equal battery, the opposite poles being to earth.

If DG, FK be the potentials at D and F, the line of fall of potential will cut AB in the middle, or the middle of

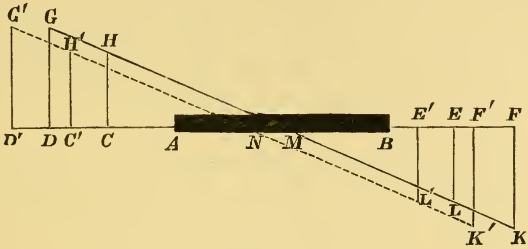

Fig. 69.

the cable is at zero-potential. The *equal* differences $DG - CH$ and $FK - EL$ can be measured at the two ends by quadrant electrometers (for instance). If now a fault arise at N, the potential at this point some time after the attachment of the batteries will come to zero, so that the differences $DG - CH$ and $FK - EL$ will no longer be the same as before, or equal. If however we now alter the variable resistances increasing AC by CC', and diminishing BE by EE' when $CC' = EE'$, it is clear that by properly choosing these resistances we shall get the new line of fall of potential parallel to the old one, and passing through N instead of M. We shall then have $NC' = MC$; $ND' = MD$; $NE' = ME$; $NF' = MF$, and we shall have the same differences of potential at the two ends as before; in fact

$$G'D' - H'C' = F'K' - E'L' = DG - CH.$$

In this case the amount of resistances added at A and subtracted at B gives the distance of the fault from the middle of the cable towards A.

If the fault be at the middle of the cable, it is clear that the result is not affected by *normal* leakage, and if it be not at the middle, allowance can easily be made for the error it produces.

195. *5th Method.* When the core of a submarine cable is broken, while the sheath remains unbroken.

The best means in this case is to measure the electrostatic capacity of the broken part of the cable; then, knowing by

previous experiment the capacity per mile, a simple division gives us the distance of the fault.

A large number of other methods are used in practice, but the principles of them all will be understood from the foregoing examples. For descriptions of these methods the reader is referred to Mr Latimer Clark's *Electrical Measurements*.

196. In our previous examples we have assumed the conductors cylindrical, and the lines of flow everywhere parallel to their length. These are the cases which most frequently occur in practice. We shall however give now two or three examples of calculating the resistance in conductors not linear in form.

Prop. VI. To calculate the resistance of a conductor bounded by two coaxial cylindrical surfaces. This will apply to the liquid in the circular form of Daniell's or Bunsen's cell.

Neglecting a portion near the ends, we shall assume the tubes of flow everywhere perpendicular to the axis of the cylinder.

Let the figure represent a section of the cylinder, and conceive it made up of concentric thin cylinders, such as PQ. The tubes of flow will be everywhere perpendicular to this cylindrical shell, and we may assume its resistance the same as for a linear conductor, whose section is its area, and length its thickness. Hence the resistance of the elementary cylinder

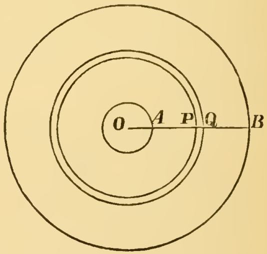

Fig. 70.

$$=\frac{1}{c}\cdot\frac{PQ}{2\pi\,.\,OP\,.\,l},$$

where c is specific conductivity, and l the length of the cylinder.

The resistance of the whole cylinder will therefore be

$$\frac{1}{2\pi cl}\cdot\Sigma\frac{PQ}{OP}.$$

But, as before,

$$\frac{PQ}{OP} = \log\left(1 + \frac{PQ}{OP}\right) = \log\frac{OQ}{OP},$$

hence resistance of elementary cylinder PQ

$$= \frac{1}{2\pi cl}(\log OQ - \log OP).$$

Summing all successive differences, we have for the resistance of the cylinder

$$\frac{1}{2\pi cl}(\log OB - \log OA)$$

$$= \frac{\log\frac{r_2}{r_1}}{2\pi cl}$$

where r_2 is the external and r_1 the internal radius of the cylinder.

This result is a particular example of the theorem of Art. 179, and might have been deduced from the capacity of the cylinder investigated in Art. 113.

197. Prop. VII. To investigate the resistance of a large solid body of any form having two electrodes connected with the poles of a battery sunk in it to a considerable depth.

This investigation of course applies to the resistance of the Earth treated as the return line in Telegraphy.

We shall represent the two electrodes as two conductors sunk in the body, and charged with equal amounts of electricity, one positive and the other negative, and on the principles enunciated in Chap. III. we shall proceed to determine the form of the equipotential surfaces and lines of force.

The form of the electrodes will be of small importance except very near them, and we shall for convenience assume them to be spheres charged with quantities $+m$ and $-m$ of electricity.

In investigating the lines of force we must in theory take account of the distribution on the surface of the body. This produces a great complication in the theory, and we have here

taken the case in which the electrodes are deeply sunk, that we may neglect this distribution; in fact, the currents near the surface will be so weak that we may neglect them entirely.

On these suppositions the potential at any point distant r_1, r_2 from the electrodes will be

$$m\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

If the mass, to begin with, be at zero-potential, and $\pm V$ the potentials at the two electrodes whose radii we will take to be ρ, then $m=\rho V$, and the whole electromotive force between the electrodes is $2V$.

To find the strength of the current, we have to consider that the flow across an equipotential surface of area σ is $cF\sigma$, where F is the resultant force, and c the specific conductivity: then the whole current-strength will be given by

$$I=c\Sigma F\sigma,$$

when the summation extends over any equipotential surface.

Now the electrodes are themselves equipotential surfaces, and we may consider the summation to take place over either electrode.

Since r_2 may be regarded as infinitely large (compared to ρ) for any point on the surface of the positive electrode, the potential over this electrode is $\frac{m}{\rho}$; on the same hypothesis the force will be $\frac{m}{\rho^2}$; and the area of the electrode is $4\pi\rho^2$;

$$\therefore\ \Sigma F\sigma=4\pi\rho^2\cdot\frac{m}{\rho^2}=4\pi m=4\pi\rho V,$$

or if E be the whole electromotive force between the two electrodes $E=2V$, and we have

$$I=2\pi c\rho E.$$

But if R be the whole resistance,

$$I=\frac{E}{R};$$

$$\therefore\ R = \frac{1}{2\pi\rho \,.\, c}.$$

Hence we see that the resistance is independent of the distance between the electrodes but varies inversely as their linear dimensions.

198. It may be interesting to notice that we might have deduced the same result by considering the plane which bisects the line joining the electrodes at right angles, this being the surface of zero-potential.

199. The result indicated in the last Article is found to agree fairly with experiments.

This investigation also shows the importance of placing the electrodes in moist ground whose specific resistance is small; otherwise the resistance offered by the first layers of the soil round the electrode may be much greater than that of the whole of the rest of the earth. When a badly conducting portion occurs at a distance from the electrodes, the principle of divided circuits shows us that it produces a very small effect indeed on the whole resistance.

200. We have shown that the amount of electricity transmitted across any section of a tube of force is proportional to a quantity c which depends only on the nature of the substance. If the substance contained in any tube be not *isotropic*, c will vary, and we shall have different current-strengths in different parts. As a consequence we shall have a charge of electricity gradually developed at the surface bounding the heterotropic parts of the tube. We now give an example of this kind.

Prop. VIII. A stratified plate composed of parallel isotropic laminæ has its opposite faces kept at given potentials, to find the amount of the electric charge at the surface bounding two layers.

Let there be three layers A, B, C, and let t_1, R_1, C_1, ρ_1, K_1 be the thickness, resistance, capacity, specific resistance, and inductive capacity respectively of A, and let similar letters with suffixes 2, 3 denote those of B, C.

Let the potentials at the surfaces A, B; B, C be initially V_1 and V_2, and finally V_1' and V_2'.

Let also the potentials at the outer surfaces be V and 0.

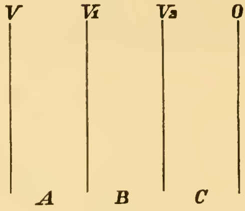

Fig. 71.

Initially, since there is no electrostatic charge on the surfaces A, B and B, C, we shall have

$$-C_1(V-V_1)+C_2(V_1-V_2)=0 \text{ at } A, B,$$

and
$$-C_2(V_1-V_2)+C_3V_2=0 \text{ at } B, C;$$

$$\therefore \frac{V-V_1}{\frac{1}{C_1}}=\frac{V_1-V_2}{\frac{1}{C_2}}=\frac{V_2}{\frac{1}{C_3}};$$

$$\therefore \text{ each of them} = \frac{V}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}=\frac{V}{\frac{1}{C}} \text{ suppose.}$$

These equations give

$$V-V_1=\frac{C}{C_1}V;\ \ V_1-V_2=\frac{C}{C_2}V;\ \ V_2=\frac{C}{C_3}V.$$

Next for the current-strength in A, B, C respectively, we have

$$I_1=\frac{V-V_1}{R_1},$$

$$I_2=\frac{V_1-V_2}{R_2},$$

$$I_3=\frac{V_2}{R_3}.$$

Hence $I_1 = I_2 = I_3$ only if

$$\frac{V - V_1}{R_1} = \frac{V_1 - V_2}{R_2} = \frac{V_2}{R_3} = \frac{V}{R_1 + R_2 + R_3} = \frac{V}{R} \text{ suppose ;}$$

or substituting for $V - V_1$ its value found above

$$\frac{C}{C_1} V = \frac{R_1}{R} V, \text{ and similarly } R_1C_1 = R_2C_2 = R_3C_3 = RC.$$

If this relation does not hold good, a gradual storing up of electricity will take place at the surfaces A, B and B, C. The law of the development of these charges is complicated, but their ultimate amount we can easily see. In fact, the storing up will go on until the currents in the three plates are equal. Hence if V_1' and V_2' be the ultimate potentials at A, B; B, C respectively,

$$\frac{V - V_1'}{R_1} = \frac{V_1' - V_2'}{R_2} = \frac{V_2'}{R_3}, \text{ and therefore} = \frac{V}{R}.$$

Under these conditions the charge bound on the plate A at the surface A, B

$$= - C_1 (V - V_1');$$

that bound on plate B at surface A, B

$$= + C_2 (V_1' - V_2');$$

and that bound on the plate B at surface B, C

$$= - C_2 (V_1' - V_2');$$

and that bound on plate C at surface B, C

$$= + C_3 V_2'.$$

Hence the whole charge on the surface A, B

$$= - C_1 (V - V_1') + C_2 (V_1' - V_2')$$

$$= - \frac{R_1 C_1}{R} V + \frac{R_2 C_2}{R} V = \frac{V}{R} (- R_1 C_1 + R_2 C_2)$$

and the charge on the surface B, C

$$= - C_2 (V_1' - V_2') + C_3 V_2'$$

$$= \frac{V}{R} (- R_2 C_2 + R_3 C_3).$$

But by Art. 179,

$$R_1C_1 = \frac{1}{4\pi}\rho_1K_1;\ R_2C_2 = \frac{1}{4\pi}\rho_2K_2;\ \text{and } R_3C_3 = \frac{1}{4\pi}\rho_3K_3.$$

Hence the amount at surface A, B

$$= \frac{V}{4\pi R}(-\rho_1K_1 + \rho_2K_2),$$

and the amount at surface B, C

$$= \frac{V}{4\pi R}(-\rho_2K_2 + \rho_3K_3).$$

The same theory might be extended to any number of plates.

It is to be noticed in the above results that the charges accumulated at the surfaces are independent of the thicknesses of the plates, and depend only on the change in value of ρK.

If the outer surfaces be brought to zero, it is clear that this accumulation within the conductor will in part be conducted back again in reverse order to the exterior, until the whole is discharged.

Cor. 1. If only some of the plates be conducting and others non-conductors, the same general effects will follow. Thus if B be a conductor, charges will be developed on the surfaces A, B and B, C, which will be bound across the dielectric to parts of the charges on A and C, which parts will by that means be disguised. The amounts so disguised the student can easily investigate for himself.

Cor. 2. It has nowhere in the above investigation been assumed that the plates are plane or bounded by plane surfaces, but only that heterotropic portions of the medium are bounded by equipotential surfaces. This will generally be the case in practice since the plates (of glass for instance) are thin and their character will vary regularly from the surface towards the interior, the variation being probably due in a large measure to unequal cooling of the internal and external parts.

201. This proposition is important, since in the opinion of Prof. Clerk Maxwell and M. Gaugain it is the best ex-

planation yet offered of the phenomenon of 'electrical absorption,' as observed in the residual charge of a Leyden jar. Faraday attributed it to a partial soaking of the electricity from opposite sides of the dielectric into its substance. This explanation contradicts Faraday's own principle of the impossibility of charging any mass of matter bodily with electricity. The explanation furnished by the stratified medium shows that the accumulation takes place after the manner of a compound condenser on strictly electrostatic principles, each charge being bound across the dielectric to an equal amount of opposite electricity. As a minor point this explanation confirms the observed fact that with air as dielectric there is no residual charge. We can easily see, however, that whatever substance except air the dielectric be composed of, owing to imperfect annealing during cooling, and the necessary irregularity in composition, the medium will not be perfectly isotropic, and where the medium is not isotropic the phenomenon of internal accumulation must arise.

202. The method of electrical images can sometimes be employed to solve the problem of conduction through heterogeneous bodies. We have shown (Art. 197) that the problem of conduction through a given homogeneous medium resolves itself into constructing systems of equipotential surfaces and lines of force due to a distribution of electricity at certain points in the medium treated at first as dielectric. Thus if in a homogeneous medium we have a single source of electricity the equipotential surfaces will be spheres. Let us suppose a quantity of electricity Q at the source, then the force over a sphere of radius R will be $\frac{Q}{R^2}$, and $\Sigma F\sigma$ over this will be $4\pi Q$. Hence (Art. 197) if ρ be the specific resistance of the substance $\frac{4\pi Q}{\rho}$ measures the quantity of electricity flowing from the source per second. If we have given that the flow of electricity from the source is I units per second or that the whole current-strength is I, the imaginary quantity of electricity at the source is $\frac{\rho I}{4\pi}$.

If we have two media of different conducting powers, we have seen (Art. 200) that there will be at first a storing of electricity at the common surface, and this distribution not being equipotential will alter the equipotential surfaces in the field. It is easy however to see that the potential must satisfy two conditions. First, at every point of the bounding surface the potential in the two media must be the same. Secondly, the quantity of electricity transmitted in one medium towards any area on the surface of separation must equal that transmitted from the same area in the other medium. If then we can find a system of imaginary sources in the media which will satisfy these conditions at every point, we can map out by means of them the equipotential surfaces throughout the whole of the two media, and solve directly the problem of conduction through them.

203. Prop. IX. To investigate the conduction of electricity through two media of different resistance separated by a plane but otherwise unbounded; the source of electricity being at a given point in one medium.

Take P the source of electricity in the medium A suppose, let P_1 be its image in the plane of separation of A and B. Suppose the potential in the medium A to be that due to E at P and E_1 at P_1, and the potential in the medium B due to E_2 at P. We must apply the criteria of the last Article to show whether these suppositions can be made to fulfil the conditions of the problem.

First. The potential at Q a point in the medium A and in the plane of separation is $\dfrac{E+E_1}{PQ}$, and the potential at the same point Q in the medium B is $\dfrac{E_2}{P_1Q}=\dfrac{E_2}{PQ}$.

Hence $$E_2=E+E_1\text{......................}(1).$$

Secondly. If ρ_1, ρ_2 be the specific resistances of the two media the quantity of electricity transmitted per second through an area σ, near Q in the medium A, is $\dfrac{1}{\rho_1}F\sigma$ (Art. 157), where F is the force perpendicular to σ. But (by Art.

114) $F=\dfrac{(E_1-E)p}{PQ^3}$, where p is the distance of the source from the plane. Hence the quantity transmitted through σ in $A=\dfrac{1}{\rho_1}(E_1-E)\dfrac{p\sigma}{PQ^3}$.

Similarly the conduction in $B=-\dfrac{1}{\rho_2}E_2\dfrac{p\sigma}{PQ^3}$;

$$\therefore \frac{E_1-E}{\rho_1}=-\frac{E_2}{\rho_2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) we have

$$E_1=\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\,.\,E,$$

$$E_2=\frac{2\rho_2}{\rho_1+\rho_2}\,.\,E,$$

which shows the original supposition admissible and gives a complete solution of the problem of conduction. If the medium A be a perfect insulator ρ_1 is infinite, and in this case $E_1=-E$, and $E_2=0$, the ordinary electrostatic problem of induction in an infinite conducting plate by an electrified particle.

Cor. 1. It may be proved by assuming the source a small spherical electrode that the flow of electricity per second from the source is $\dfrac{4\pi E}{\rho_1}$.

Cor. 2. The quantity conducted per second from the medium A to the medium B across the plane of separation is found by an easy summation to be $\dfrac{4\pi E}{\rho_1+\rho_2}$.

Cor. 3. This gives the solution of conduction through two media bounded by a plane for any system of sources whatever, since the potential at any point will be simply that due to the superposition of the potentials of the sources and of their imaginary images as given above.

Cor. 4. If θ_1, θ_2 be the inclinations of the line of force

on opposite sides of the plane of separation of A, B to the normal to that plane, it is at once seen that

$$\tan\theta_1 = \frac{E+E_1}{E-E_1}\,.\tan\theta_2 = \frac{\rho_2}{\rho_1}\tan\theta_2,$$

or

$$\rho_1\tan\theta_1 = \rho_2\tan\theta_2,$$

which is sometimes called the law of refraction of a line of force from one medium into another.

204. The same method may be applied to the explanation of Nobili's rings when the fluid stratum is very thick compared to the distance of the electrode from the metal plate.

These rings, as is well known, are produced when a wire forming one electrode is immersed in a fluid spread over a metal plate forming the other electrode, these electrodes being connected with a battery. If the fluid is an electrolyte it will be decomposed, and some product of decomposition will be laid in thin layers on the metal plate. The varying colour produced by thin plates according to their thickness will vary with the thickness of this deposit, and the thickness of the deposit will depend at each point on the intensity of the electrical force.

Since the resistance of the liquid is always very great compared with that of the metal sheet we may neglect ρ_2 compared with ρ_1 and we have $E_1 = -E$, when the problem reduces to that of the ordinary electrical image investigated in Art. 114.

The force at each point on the plate will be that due to the electrified point combined with its electrical image, and is shown to vary inversely as the cube of the distance between the electrode and the point on the plate. Hence the thickness of the deposit also varies as the inverse cube of the distance.

EXAMPLES ON CHAPTER VII.

The following is a list of the specific resistances (see Art. 161) of various substances at 0° C., referred to the B.A. unit. ("*Numerical Tables and Constants*" by S. Lupton.)

(1) Specific resistances of metals in microhms (10^{-6} ohm), to which are

added in second column the percentage variation per 1° C. according to Dr Matthiesen (Ayrton's "*Practical Electricity*").

Silver, annealed	1·521	·377
Copper, hard drawn	1·642	
,, annealed	1·590	·388
Platinum, annealed	9·158	
Iron, soft	9·827	·5 (about)
Mercury	96·146	·072
Bismuth	132·65	·354
German silver	21·17	·044
Brass	5·8	

(2) Specific resistances of liquids in ohms:

Water at 4° C.		$9 \cdot 1 \times 10^6$
,, 11° C.		$3 \cdot 4 \times 10^5$
Dilute hydrogen sulphate	5 % acid at 18° C.	4·88
,, ,,	20 % ,, ,,	1·562
,, ,,	30 % ,, ,,	1·38
,, ,,	40 % ,, ,,	1·5
Hydrogen nitrate at	18° C.	1·61
Copper sulphate *sat.* at	10° C.	29·3
Zinc sulphate *sat.* at	14° C.	21·5
Sodium chloride *sat.* at	13° C.	5·3.

(3) Specific resistance of insulators in megohms (10^6 ohms).

Glass (crystal) below	40° C.	infinite
at	46° C.	$6 \cdot 182 \times 10^9$
at	105° C.	$1 \cdot 16 \times 10^7$
Paraffin at	46° C.	$3 \cdot 4 \times 10^{10}$
Ebonite at	46° C.	$2 \cdot 8 \times 10^{10}$

1. Find the length of copper wire ·5 mm. in diameter whose resistance is 1 ohm. *Ans.* 11·95 metres.

2. Find the length of a platinum and of a German-silver wire, of the same gauge as in question 1, which has 1 ohm resistance. *Ans.* 2·143 metres.

3. What must be the relation in diameters of a copper and iron wire of which equal lengths give equal resistances? *Ans.* ·41 : 1.

4. Siemens' unit of resistance is the resistance of a column of mercury 1 metre long whose section is a square mm. Compare it with the ohm. *Ans.* 1·08 ohm.

5. A standard coil of German-silver is issued measuring 1 ohm at 6° C. Find its resistance at 20° C. and at 50° C.

6. An annealed copper ribbon has a resistance of 21·96 ohms at 15° C. Find its resistance at 0° C.

7. Given the specific resistance (ρ), the specific gravity (s), the length in cm. (l), and the mass in grams (m) of a wire, show that its resistance is given by $\dfrac{\rho s l^2}{m}$.

8. Find the resistance of a copper wire 10 metres long which weighs 4·45 grams (S.G. of copper = 8·9). Compare it with the resistance of a platinum wire having the same weight and length (S.G. of platinum = 22·1).

Ans. 3·284 ohms: 1 to 13·8.

9. Given that 1 cm. = ·3937 in., show that the resistance of wire whose length (l) and section (s) are given in inches is $\cdot 3937\,\dfrac{\rho l}{s}$; ρ being the specific resistance given above.

10. Find the resistance of a hundred miles of iron telegraph wire whose diameter is $\frac{1}{5}$ inch.

Ans. 781 ohms (nearly).

11. If 100 in. of copper wire weighing 100 grs. has resistance ·1516 ohm, find the resistance of 50 in. weighing 200 grs. *Ans.* ·01895 ohm.

12. Two cells, each 1 ohm internal resistance, are connected in compound series with a wire whose resistance is 1 ohm. If each of these, when connected singly by stout wires to a galvanometer* of no appreciable resistance, deflect it 25°, how much will the combination deflect it? *Ans.* 17° 15′.

13. A single thermo-electric couple deflects a galvanometer of 100 ohms resistance 30′, how much will a hundred such couples in compound series deflect it? (The resistance of the couples themselves may be neglected.) *Ans.* 41° 7′.

14. The internal resistance of a cell is half an ohm; when a galvanometer of one ohm resistance is connected with it by short thick wires it is deflected 15°: by how much will it be deflected if for one of the thick wires a wire of 1·5 ohms resistance be substituted? *Ans.* $7\frac{2}{3}$°.

15. A cell of $\frac{1}{3}$ ohm resistance deflects a galvanometer of unknown resistance 45°, the connection being made by short

* The galvanometer, unless the contrary be stated, may be assumed to be a tangent galvanometer.

stout wires. If a wire of 3 ohms resistance be substituted for one of the stout wires the deflection is 30°. Find the resistance of the galvanometer. *Ans.* 3·8 ohms nearly.

16. A galvanometer of no sensible resistance is deflected 45° by a cell connected with stout wires. When a resistance of 5 ohms is introduced, the deflection sinks to 30°; find the resistance of the cell. *Ans.* 6·8 ohms.

17. A Bunsen and a Daniell cell are placed in the same circuit, first with their electromotive forces in the same direction, and secondly in opposite directions, the deflections being respectively 30°·2 and 10°·6. Compare their electromotive forces. *Ans.* Bunsen's cell = 1·9 of Daniell's.

18. If an insulated closed voltaic circuit be connected at a point whose potential is V with an insulated conductor whose capacity is C', show that the potential at this point becomes $\frac{CV}{C+C'}$, where C is the capacity of the original circuit; and that there will be a fall of potential through the whole circuit equal to $\frac{C'V}{C+C'}$.

19. The terminals of an insulated battery of 520 cells are united by 78 miles of cable which have the same resistance as the battery. At the extremity of the cable, next the zincode, 43 miles of cable are connected with the circuit at one extremity, the whole being insulated. Show that the potential at the zincode immediately falls in the ratio 121 to 78. The capacity of the battery is neglected.

20. A battery of seven cells has its ends joined by a wire whose resistance is three times that of the battery. At the junction of the third and fourth cells there is connection with the earth. Draw a diagram of the fall of potential in the circuit. How will the current in the circuit be affected?

21. Three Daniell's cells are arranged in compound circuit with a resistance of ten times one cell between each two. Calculate the current when the terminals are joined by a stout wire, and draw a diagram of the fall in potential through the circuit.

22. If the difference of potential between the two terminals of a battery be measured when the circuit is open, and if the same difference of potential, measured when closed, is one-half its former value, show that the external and internal resistances are equal.

23. If by introducing into a circuit (formerly closed by a short stout wire) a certain measured resistance, the current-strength sink to one-half its former value, show that the resistance introduced is equal to the internal resistance.

24. The cells of a battery made of square metal plates whose edge is 8 cm., are separated by 1·5 cm. of hydrogen sulphate (dil. 20 % acid). Find the internal resistance of each cell. *Ans.* ·0366 ohm.

25. Six cells of Ques. 24 each of E.M.F. 1 volt are arranged in series and have their terminals joined by a copper wire 10 m. long and ·5 sq. mm. in cross-section. Find the current transmitted and the rise in temperature of the wire in five seconds, supposing no heat to escape, and neglecting the change in resistance of the wire owing to increase of temperature. (In copper, specific gravity = 9 and specific heat = 0·1.) *Ans.* 10·9 amps.: 10°C. nearly.

26. Find also the amount of heat developed in the battery under the same conditions as in the last question. *Ans.* 31 gm-degrees.

27. A line joining two places A, B, 130 miles apart, at 10 miles from A drops from its support and rests on another wire which makes earth at distances 30 and 40 miles. Find the ratio of the current-strengths at A and B.
Ans. If sent from A, 8 to 1, if from B, 12 to 19.

28. In a closed circuit, two points are joined by a conductor of given resistance. Given the resistances, write down equations to determine the currents in all the branches.

29. Two cells AA_1, BB_1, are simple-circuited by wires ACB, A_1DB_1, and the points C, D joined by a wire. Given the resistances and electromotive forces, find the current in CD.

30. Find the current in the preceding question, supposing the two cells arranged in series.

31. Show in the last question if the current in CD vanish, and E_1, E_2 be the electromotive forces in AA_1, BB_1 respectively, then

$$E_1 : E_2 :: \text{resistance in } CAA_1D : \text{resistance in } CBB_1D.$$

Devise an experimental method for comparing E.M.F.'s.

32. Four cells, the resistance in each being 3 ohms, are used with external resistance 10 ohms; will it be better to use them in a simple or compound circuit?

Ans. Compound circuit.

33. Find the smallest external resistance in the preceding question with which it will be an advantage to use a compound circuit. *Ans.* 3 ohms.

34. When the poles of 100 cells in compound circuit are joined by a thick wire, a galvanometer deflects 60°; when 100 ohms are inserted the deflection sinks to 30°. Find the internal resistance of one cell. *Ans.* ·5 ohm.

35. Of two cells one is short-circuited and gives 60° deflection, and on introducing 6 ohms the deflection becomes 45°: another, when short-circuited, gives 45°, and on introducing 6 ohms sinks to 30. Find the ratio of their electromotive forces. *Ans.* $\sqrt{3}$ to 1.

36. The zincode of a battery of 100 cells is to earth, and the other end communicates with the end A of a line AB, whose distant end B is to earth. The resistance of the line AB is ten times that of the battery. If now a second battery of 50 cells having also its zincode to earth have its other end (as well as that of the former battery) to the end A of the line, find the change in the current in the line.

Ans. Current is $\frac{22}{31}$ of former value.

37. A battery of 20 ohms resistance sends a current through a galvanometer of 15 ohms resistance to a line of 70 ohms resistance, and at the other end is a galvanometer of 15 ohms resistance. What effect is produced on each galvanometer if there be a fault whose resistance is 20 ohms in the middle of the line?

Ans. Current in battery galvanometer is altered in ratio 84 to 59, that in line galvanometer in ratio 24 to 59.

38. A telegraph wire having a battery and galvanometer at the sending, and a galvanometer at the receiving end, when in good insulation transmits a current I. After a fault has arisen in the line, the current at the sending end rises to I_1, and at the receiving end sinks to I_2; show that the fault divides the whole resistance in the circuit in the ratio $I - I_2$ to $I_1 - I$.

39. In a compound arrangement with 3 similar cells and no external resistance except a galvanometer the deflection was observed to be 60°. Using one cell only and the same external conditions the deflection was 44°. On introducing into the latter arrangement 20 ohms additional resistance the deflection sank to 25°. Find the resistance of the galvanometer and of each cell of the battery.

Ans. Internal resistance 6 ohms.
Galvanometer „ 12·5 ohms.

40. One hundred cells each of internal resistance 4 ohms are to be used with 25 ohms external resistance. Find the arrangement which will give the strongest current and the strength of this current.

Ans. 4 rows of 25 cells. Current-strength $\frac{1}{2}E$.

41. What is the best arrangement of 6 cells each of $\frac{2}{3}$ ohm resistance against an external resistance of 2 ohms?

Ans. 6 cells in series, or 2 rows of 3 cells.

42. What is the best arrangement of 20 cells each of 8 ohms resistance against an external resistance of 4 ohms?

Ans. 4 rows each of 5 cells.

43. A battery of three cells is arranged in a mixed circuit so that there are two rows containing 1 and 2 cells respectively, and the terminals are connected by a wire of resistance R. Find the current-strength.

Ans. $\dfrac{4E}{3R + 2r}$, where E is the electromotive force and r the resistance in each cell.

44. A battery of six cells is arranged in mixed circuit so that there are three rows containing respectively 1, 2 and

3 cells. Find the current-strength in a conductor joining the terminals.

Ans. $\dfrac{18E}{6r+11R}$.

45. A battery is arranged in mixed circuit consisting of n rows containing respectively 1, 2, 3......n cells. Show that the current-strength in a wire joining the terminals is given by

$$\frac{nE}{r+RS},$$

where $$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots + \frac{1}{n}.$$

46. In Wheatstone's bridge as commonly used (Fig. 64), show that when the branch BD is open the difference of potential between B and D is given by

$$\frac{E(ps - rq)}{R(p+q+r+s)+(p+q)(r+s)}.$$

47. ABC is a triangle formed by straight and uniform conductors, and OA, OB, OC are similar conductors joining O to the angular points; find the condition that OA may be conjugate to BC.

48. Show that in the preceding question if O be a point such that OA and BC are always conjugate, the locus of O is a circle.

49. Show also that there is one and only one point such that OA is conjugate to BC, OC to AB and OB to AC.

50. A current of one ampere passes through a copper wire of 1 sq. mm. section. Find the rise in temperature per minute supposing no escape of heat (see question 25).

51. Find the section of a copper rod which will transmit 100 amperes without rising in temperature more than one-tenth of a degree C. per minute.

52. In a wire joining the poles of a galvanic cell of small resistance the wire is more heated if it be of copper than if it be of platinum of the same dimensions, but if the internal resistance be large, the platinum wire will be more heated than the copper. Explain this.

53. Show in the previous question that the heat given out in the two wires would be equal, if the internal resistance of the cell be a geometric mean between the resistances of the copper and platinum wires.

54. If a chain made of alternate links of platinum and silver have a strong current sent through it the platinum becomes red-hot while the silver remains cold and black. Explain this.

55. If part of a loop of wire which is rendered red-hot by the passage of a current be held in a spirit flame, the part outside the flame becomes black, but if the same part be dipped in water the part outside immediately glows with a redder light. Explain this.

56. An electrode is sunk in the centre of a spherica mass of matter whose specific resistance is k_1, which is surrounded by an infinite mass of specific resistance k_2. Find the potential at any point in the mass. Discuss the effect on the current when k_1 is very large compared with k_2 and apply it to show the effect of a 'bad earth' in Telegraphy.

57. Show that in a circuit in which the galvanometer resistance is very large compared to all other resistance the galvanometer reading will not be diminished by introducing a shunt.

CHAPTER VIII.

MAGNETISM.

205. We have here a new class of phenomena to consider. They are exhibited most strongly in the varieties of iron, though probably in some degree in almost all bodies. We shall place before the reader a brief sketch of the phenomena with which we assume he is already familiar, and develop as far as possible step by step the theory deduced from them.

Experiment 1. A piece of magnetic iron or a bar of steel magnetized is found to exercise a peculiar force on pieces of iron. This force vanishes near the middle of the bar, and increases in magnitude very rapidly towards the ends. The force is often spoken of as resident in the ends of the magnet, which are therefore called its *poles.*

As in electricity the force is often attributed to an imaginary distribution of *magnetic fluid* over the poles. This must of course be treated as only an image representing to our minds the existing force.

206. *Experiment* 2. If the two poles of any magnet A be brought in succession into the neighbourhood of a pole of a second magnet B, one will suffer an attractive and the other a repulsive force. If another magnet C be taken, and the poles of A and C, which are both attracted or both repelled by one pole of B, be made to act on each other, there will be a repulsive force between them. If again two poles be taken, one of which is attracted and the other repelled by either pole of B, there will be an attractive force.

We infer from these experiments that *every magnet has two dissimilar poles, and that like poles repel each other, but unlike poles attract each other.*

The nomenclature of these poles is derived from the behaviour of the magnet when suspended about its centre of gravity. Each magnet, if free from other magnetic influence, then points in this country nearly due N. and S. The end that points northwards is called the north pole, and that which points southwards the south pole.

We shall however speak of the magnetism distributed over the north pole of a magnet as *positive*, and that over the south pole as *negative* magnetism.

207. *Experiment* 3. The forces exerted by the two poles of a magnet on any third pole are always equal in magnitude, though opposite in direction. This will be conveniently expressed by saying that if the strength of one pole be $+m$, that of the other is $-m$.

The strength of a pole can be expressed by the force it exerts on another pole, and the unit in terms of which it is measured can be defined thus:

DEF. A UNIT MAGNETIC POLE *is a pole which exerts a unit of force* (*or a dyne*) *at unit distance on another equal pole.*

As in statical electricity, we here define Magnetic Density.

DEF. MAGNETIC DENSITY *at any point on the surface of a magnetized mass is the quantity of magnetism per sq. cm. of surface separated at that point.*

We add here the definition of the Moment of a Magnet, a term we shall have often occasion to employ.

DEF. MOMENT OF A MAGNET *is the product of the strength of either pole into the distance between the poles.*

We shall assume that in a compound magnet, where the axes of all the elementary magnets are in the same direction, the moment of the whole magnet is the sum of the moments of the elements. This will appear true when we come to the physical meaning of Magnetic Moment.

208. *Experiment* 4. The force between two magnetic poles is found to vary as the product of their strengths when at the same distance, and inversely as the square of their distance when the distances are varied.

Thus if we have two poles whose intensities are m and m' at a distance r from each other, the force between them is $\frac{mm'}{r^2}$, this force being repulsive when the numerator is positive, and attractive when the numerator is negative.

This experiment shows that round every distribution of magnetism a field of magnetic force exists whose general laws will be identical with those of gravitational force discussed in Chapter II. The only change required will be to read unit magnetic pole or particle charged with unit of positive magnetism, for the unit of mass there employed to test the field. Our definitions will then stand thus:

1. *Field of Magnetic Force* is the medium surrounding magnetized bodies within which work has to be done to move a magnetic pole. We are not at liberty to assume as in electricity that the field of force does not extend within the magnetized mass itself, since there is no experiment to show that within a magnetized mass the magnetic force vanishes.

2. *Lines of Force* are lines in the field such that the tangent at each point shows the direction in which a magnetized particle placed there would be urged. Since a positively and negatively magnetized particle will be urged in opposite directions along the line of force, it is convenient to define the positive direction of the line of force as that in which a positively magnetized particle would be urged.

3. *Strength of field at a point, or Magnetic Force at a point,* is the force with which a unit magnetic pole would be urged if placed at that point.

4. *Magnetic potential at a point* is the work which would be done in bringing a unit magnetic pole to that point from an infinite distance or out of the field of magnetic force. If there be a distribution of magnetism consisting of quantities $m_1, m_2 \ldots$ at distances $r_1, r_2 \ldots$ from the given point, the measure of the magnetic potential at the point will be

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots\ldots = \Sigma \frac{m}{r}.$$

209. We may of course apply in the magnetic field the characteristic property of a tube of force, viz. that through it $F\sigma$ is a constant, F being the average force per plus unit or strength of field at right angles to the small area σ. We follow Faraday in representing this proposition on the magnetic field in a different way. Suppose one equipotential surface in the field so mapped out that the number of lines of force arising from any closed area on it shall be proportional to the product of the area and the average force over it, so that the force at any point on that surface is proportional to the number of lines of force per unit area of surface near that point. It will be seen that we do not here limit the number of lines of force, but only the closeness of their distribution. Thus a certain density of distribution representing unit force, a distribution of twice as many per unit area or twice the density represents a force whose numerical measure is two units, and so on. It is usual to make the further convention that the density or closeness of lines of force where the force is unity shall itself be represented by one unit, so that the force at any point is measured by the number of lines of force per unit area near that point. With the above limitations $F\sigma$ measures the number of lines of force which intersect an element σ of the equipotential surface, over which the force is F, and the same number must intersect all elements of all equipotential surfaces made by the same tube of force. Since $F\sigma$ is constant throughout the tube it follows that the force at a point on any other equipotential surface will be measured by the number of lines of force per unit area near that point. The reasoning of Art. 43 can be applied to prove that for any small area inclined to an equipotential surface, the number of lines of force per unit area is the numerical measure of the force resolved perpendicular to that area. We thus arrive at the proposition that at any point in a magnetic field and in any direction through that point the component of the strength of field is the number of lines of force per unit area round that point, cutting through a surface at right angles to the given direction.

To connect the lines of force with magnetic density let us consider an isolated pole of strength m, the strength of field

at distance r from it is $\frac{m}{r^2}$, this must therefore be the strength of field everywhere over an equipotential surface of radius r. But the area of this surface is $4\pi r^2$ and, since the lines of force must number $\frac{m}{r^2}$ per unit area, the total number of lines of force due to the pole m must be $4\pi m$, which is equivalent to saying that there are 4π lines of force for every unit of magnetism.

Since lines of force proceed from positive to negative magnetism it is more correct to say that 4π lines of force exist between a positive and negative unit of magnetism. If we have a simple magnet with pole of strength $\pm m$, the measure of the number of lines of force from one pole to the other is $4\pi m$.

When we speak of a magnetic density ρ, we mean that a distribution of free magnetism has ρ units of magnetism per unit area. In accordance with our new convention this means that from every distribution of magnetism of density ρ $4\pi\rho$ lines of force proceed from unit area.

210. *Experiment* 5. If a straight bar magnet AB, such as was referred to in preceding experiments, be broken in half as at C, we do not get two bars, one AC charged entirely with north magnetic fluid, and BC entirely with south magnetic fluid; but at C two new poles are developed on opposite sides of the plane of division, so that we get two magnets with poles of the same intensity as the original magnet.

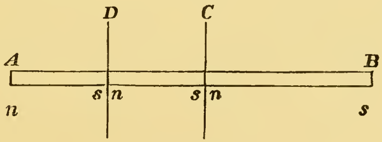

Fig. 72.

If AC be broken at D, we find again two new poles developed at the place of fracture, and this operation may be repeated indefinitely, each fragment broken off still being magnetized similarly to the given magnet.

This leads us to the conception of a magnet as made up of molecules, each of which is a magnet, the resultant magnet being due to the combined action of all the elementary magnets of which it is composed.

This experiment shows that it is only for parts of the field outside the magnet that we can consider the magnetism as a surface distribution. In attempting to apply the property of a field of force (Art. 46) that in passing through a sheet of attracting matter the product $F\sigma$ changes by $4\pi\rho\sigma$, where $\rho\sigma$ is the quantity of attracting matter intercepted, we must consider that in passing through any sheet of the magnetized substance, however thin, we really pass through equal and opposite quantities of attracting matter. We are therefore bound to consider $F\sigma$ as having the same value within the magnetized surface that it has outside it. We must therefore think of the lines of force as suffering no discontinuity but forming closed curves proceeding from North to South outside and from South to North inside the bar.

We can arrive at the same result in perhaps a more convincing way by imagining a thin slice cut from a magnet such as PQ, and considering the magnetic field within the cavity.

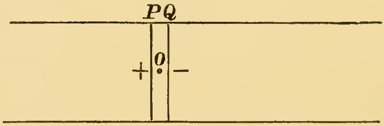

Fig. 73.

If $\pm\rho$ be the density on the ends of the bar (not shown), supposing the right hand the north and left hand the south pole, we shall have on surface P a density $+\rho$ and on surface Q a density $-\rho$.

The force on a plus unit at O will be (Art. 36) $2\pi\rho$ due to the distribution on P and $-2\pi\rho$ due to the distribution on Q, the two compounding into a force $4\pi\rho$ from P to Q. We conclude that the narrow cavity is traversed by $4\pi\rho$ lines of force per unit area in direction from the south to the north pole of the magnet, if we neglect the effect of the free distributions on the ends of the magnet.

211. If a small rectangular parallelepiped having its length in the direction of magnetization were removed from the mass of the magnet without disturbing its magnetization, the density of magnetism on its ends would measure the intensity of the magnetization near that part of the mass. Suppose one of these small magnets of length s and section α to have a distribution over its end whose density is ρ, the strength of its poles is therefore $\pm \rho\alpha$, and its magnetic moment (Art. 207) is $\rho\alpha s$. But αs is the volume of the small magnet, and therefore ρ may be defined by the quotient of the magnetic moment of the magnet by its volume. This quotient within the mass (where there is no free magnetism) is called the intensity of magnetization. If the magnet be not uniformly magnetized, we may still at a particular point compute the magnetic moment per unit volume, and define this to be the intensity of magnetization at the point in the magnet.

DEF. THE INTENSITY OF MAGNETIZATION *at any point in a magnetic mass is measured by the magnetic moment per unit volume of a mass of the magnetized matter very near to the point.*

We now infer that at any point in a magnetized mass at which the intensity of magnetization is μ there exist $4\pi\mu$ lines of force per unit area at right angles to the direction of magnetization. If however a plane area be taken whose normal makes an angle θ with the direction of magnetization the lines of force per unit area intercepting this plane will be smaller in the proportion of 1 to $\cos\theta$, or the number per unit area becomes $4\pi\mu \cos\theta$, or what is the same thing, the density of the distribution becomes $\mu\cos\theta$.

The same will be true if the normal to the free surface of a magnet makes an angle θ with the direction of magnetization; the density of the surface magnetism is $\mu\cos\theta$.

This leads us to a convenient mental representation of a uniformly magnetized mass of any form as far as points outside the mass are concerned. Suppose the mass M everywhere permeated by two imaginary fluids, one having density ρ and the other $-\rho$. In the neutral state of the body

these fluids are exactly superimposed and coincide, filling the whole body. In the magnetized body we may assume that the positive fluid has been moved *en masse* by simple translation through some very small space s relatively to the negative fluid. The action of the magnetized mass will in every respect agree with that of the equal masses $+M$ and $-M$, the direction of displacement being the direction of magnetization and the amount of displacement determined by the formula $\rho s = \mu$, the intensity of magnetization. For these two fluids neutralize each other throughout the body, leaving only a surface distribution whose depth is $s \cos\theta$ at a point where the normal to the surface makes an angle θ with the direction of displacement. The quantity per unit area of this fluid will be therefore $\rho s \cos\theta = \mu \cos\theta$; which also represents the density of free magnetism on a uniformly magnetized mass.

212. *Experiment* 6. A magnetic pole induces in a piece of soft iron near it a separation of the magnetic fluids; on the parts nearest to it inducing a distribution of magnetism of opposite sign, and on the parts more remote a distribution of magnetism of like sign with itself.

This induction takes place whether bodies be already magnetized or not, but its amount depends very much on the nature of the body. Thus a piece of soft iron placed in the magnetic field, becomes temporarily a strong magnet, while a piece of hardened, or cast iron, or hardened steel will not be so powerfully magnetized.

On removing the soft iron from the magnetic field, it is found at once to become very nearly neutral, retaining to a very slight extent the magnetism it had acquired under the influence of the magnetic field. It is in consequence when in the field said to be a temporary magnet.

On removing from the field the hardened iron or steel, it is found to retain to a much greater extent the magnetism induced in it, though its temporary magnetic properties were much weaker than those of the iron similarly placed. Thus the steel under magnetic induction becomes a permanent magnet. The amount of magnetism which becomes per-

manent in the magnet, can be very much increased by setting it in a state of violent vibration when in the field.

This behaviour of hardened iron and steel is explained by the existence in them of a coercitive force which is absent in the soft iron—a force causing the molecules of the body not to yield so readily to magnetic forces, and therefore not to acquire magnetism so easily, but having once acquired it to retain it for ever.

It is generally assumed that the magnetism induced in a thin rod of iron or steel is proportional to the strength of the magnetic field: this is true only within certain limits, since all known magnets reach a point of saturation after which they are unable to take up more magnetism. The law of direct proportionality only holds until the magnetism has reached about one-half the saturation charge.

The only bodies except the varieties of iron which exhibit magnetic properties when placed in a weak magnetic field are nickel and cobalt, though it has been found by Faraday, that all bodies become magnetic when placed in a field of sufficient strength. He found also in the course of these experiments that a class of bodies of which bismuth is the type have magnetic properties exactly the reverse of iron, apparently acquiring under magnetic induction a pole of the same name next the inducing pole. To these bodies he gave the name diamagnetics, calling bodies which resemble iron in their properties paramagnetics.

213. If there be in the magnetic field a small filament of iron whose length is along a line of force at a point where the strength of the field is H, the quantity of free magnetism on its ends may be represented by $\pm k \,.\, H \,.\, \alpha$ where α is the area of section of the filament, and k a constant depending on the nature of the iron, being greatest for soft iron and least for steel. k is then called the coefficient of magnetization, and is large for iron and moderate for nickel and cobalt, but only a very small fraction for all other paramagnetics, and a very small negative fraction for all diamagnetics.

DEF. THE COEFFICIENT OF MAGNETIZATION *for a given kind of iron is measured by the density of magnetism on the ends of a prism of the iron placed along the lines of force in a field of unit strength.*

214. The general problem of the magnetization of a given body by given magnetic forces is highly complex. We may point out however that the effect of bringing into a magnetic field a mass of iron will be generally greatly to increase the lines of magnetic force in that part of the field, and we may make an approximate estimate of the proportionate increase.

Let us consider the magnetic field within a mass of iron whose coefficient of magnetization is k. As in Art. 210 conceive a thin slice cut out from the magnetized mass at right angles to the lines of force close to a point at which the strength of field was H. Each filament removed was magnetized along the lines of force to a density Hk, and the cavity has a distribution on its surfaces whose density is $\pm Hk$. Hence the number of lines of force within the cavity will owing to the presence of the iron be increased by $4\pi Hk$, and the number of lines of force per unit area within the cavity becomes

$$H + 4\pi Hk = H(1 + 4\pi k).$$

The coefficient $1 + 4\pi k$ which gives the ratio in which the presence of iron in the field increases the number of lines of force is called the coefficient of magnetic induction, but now more frequently the *permeability* of the iron.

Thus if H be the number of lines of force per unit area in a field of force of air, and B the number of lines of force per unit area when iron replaces the air, then the ratio $\frac{B}{H}$ is the permeability, which is often denoted by the letter μ.

DEF. *The permeability or coefficient of magnetic induction of a substance is the ratio of the number of lines of force traversing the substance to those which would traverse the same space if air were substituted for the material.*

The permeability of air is always unity and that of iron generally a very large number, while it is moderately large for nickel and cobalt, but very near unity for all other paramagnetics. For diamagnetics k is a very small negative fraction and the permeability therefore a proper fraction very near to unity.

215. We have said that Faraday proved all substances to be more or less subject to magnetic induction. This leads us to think of the whole field of magnetic force as composed of magnetic materials differing from each other only in degree of susceptibility to magnetism. Also we have the characteristic property of the magnetic field that if the magnetizing Force at any point be H and a substance whose permeability is μ be placed there, μH is a constant through every magnetic circuit. It is easy to see that this leads to free magnetism at the surface bounding heterogeneous media. Since if μH is constant and μ changes abruptly H changes abruptly too, and this can only take place when there is a free magnetization of a surface.

This appears at once if we remember that through every tube of force $H\sigma$ is constant and σ, the area of the tube of force, can undergo no abrupt change.

If H_1, H_2 be the strengths of field on opposite sides of a surface, the density ρ of free magnetism is given by

$$H_2 - H_1 = 4\pi\rho \quad \text{.....................(1).}$$

Also if μ_1, μ_2 be the permeabilities of the materials

$$\mu_1 H_1 = \mu_2 H_2 \quad \text{.......................(2).}$$

These equations enable us to determine two of the quantities $H_1 H_2 \rho$ when the other two are given.

216. It is easily seen that the expression μH for the magnetic induction, or flux of lines of magnetic force across unit area of a magnetic circuit, is analogous to the expression cF (Art. 157) for the flow of current across unit area of a

conductor. While the equation (2) of Art. 215 for magnetic induction across the junction of two different media with a development of free magnetism is easily seen to be identical with the condition for steady current across a junction in the stratified condenser of Art. 200 given at the top of p. 207. Similarly to every problem in electric conduction corresponds one in magnetic induction if we substitute strength of magnetic for strength of electric field, permeability for conductivity and magnetic flux for current strength. This analogy we shall return to again under electromagnetism.

217. We give some cases of magnetization worth special notice.

Prop. I. To investigate the potential at any point of a straight thin cylindrical bar placed in a uniform magnetic field with its length parallel to the lines of force.

In this case the whole bar is a tube of force, and the intensity of magnetic separation is the same everywhere along it. We may conceive it made up of rows of molecules in which the magnetic separation is represented thus:

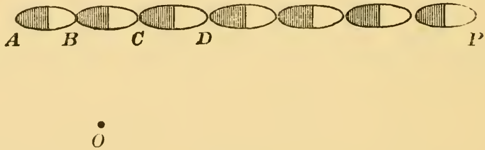

Fig. 74.

The only free magnetism acting on an external unit of magnetism at O, will be the free magnetism at the ends A and P.

If the amounts at A and P be $\pm m$ the magnetic potential at O is represented by

$$m\left(\frac{1}{OA}-\frac{1}{OP}\right)=m\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

The same will be true for all the rows of molecules, and we have for the potential of the bar on any external point O

$$\Sigma m\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

If the bar be long and very thin, r_1, r_2 will be sensibly constant over the respective ends, and we have

$$V=M\left(\frac{1}{r_1}-\frac{1}{r_2}\right),$$

where M is the strength of the pole.

This distribution of magnetism is approached practically in the bar magnet. In all bar magnets there is a certain small force at a distance from the poles which is traceable to the falling off in strength of the magnetic separation in the molecules as we go towards the ends.

Cor. If the magnet be bent round so that it forms any closed curve, the potential on any external point vanishes; for then $r_1=r_2$.

218. Prop. II. To find the potential at any external point of a thin magnetic shell in which the magnetization is everywhere normal to its surface. This is called a lamellar distribution of magnetism.

Such a shell may be conceived as made up of an infinite number of thin short magnets, placed side by side. Let the figure represent such an element, AB being the normal, σ the area of each pole, A being the north and B the south pole of the small magnet. Let O be the external point, at which we will suppose a unit pole.

Join OA and OB, and draw AN perpendicular to OB. Also describe a cone whose vertex is O and base A, and conceive two sections of this cone, one P by a sphere of radius unity, and the other Q* by a sphere of radius OA.

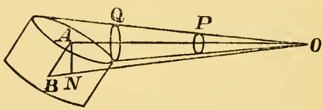

Fig. 75.

* To prevent confusion Q is made a section near the edge of the shell, but in the reasoning the plane of Q is supposed to pass through A.

Let the density of the magnetism on A and B be $\pm\rho$, the strength of each pole of the elementary magnet AB will then be $\pm\rho\sigma$.

The potential at O due to the element of the shell will then be

$$\rho\sigma\left(\frac{1}{OA}-\frac{1}{OB}\right)=\rho\sigma\frac{OB-OA}{OA\,.\,OB}=\rho\sigma\frac{BN}{OA^2}$$
$$=\frac{\rho\sigma\,.\,AB\cos\epsilon}{OA^2},$$

where ϵ is the angle ABO.

Now since the area σ is perpendicular to AB, and the area Q to OA, the angle between σ and Q must be ϵ, and Q may be regarded as the orthogonal projection of σ.

Hence $$Q=\sigma\cos\epsilon,$$

$\therefore$ the potential becomes

$$\frac{\rho\,.\,AB\times\text{area }Q}{OA^2}.$$

Again, by similar figures,

$$\text{area }Q:\text{area }P::OQ^2:OP^2$$
$$::OA^2:1,$$

since P is on a sphere of radius unity.

Hence the potential at $O=\rho AB\times\text{area }P$.

The area P is the projection by a cone of the edge of the elementary shell on a sphere of unit radius, and this may be defined as the spherical measure of the solid angle subtended at O by the shell-element. We will denote this measure by ω. If the product $\rho\,.\,AB$ is constant over the whole shell, of which AB is an element, the shell is said to be a simple magnetic shell, and this product is defined to be its strength. We shall denote it by the symbol j. It is obviously the magnetic moment per unit area. The potential of the whole shell may then be written $\Sigma j\omega$, or if the magnetic shell be simple, this reduces to $j\Sigma\omega$, and the solid angles subtended by the elements can be simply added, and will give on summation the solid angle subtended by the whole shell.

Hence the potential due to a simple magnetic shell of strength j at an external point O will be $\pm j\Omega$, where j is the strength of the shell, Ω the solid angle subtended at O by its edge, the positive sign being given when O *faces* the positive surface of the shell.

219. Prop. III. To find the potential of a magnetic field on a given simple magnetic shell placed anywhere in it, or to find the work done in carrying the shell from an infinite distance up to the given position in the field.

The expression (Art. 218)

$$\frac{\rho\sigma AB\cos\epsilon}{OA^2}$$

gives the work done in carrying a unit pole up to the given position in the field against the repulsion exercised by the shell element. If the strength of pole were m the work done would be m times as much, and by the third law of motion this equals the work done in carrying up the shell element to the given position in the field of the magnet pole. This then may be written

$$=\rho AB\,.\,\frac{m}{OA^2}\cos\epsilon\,.\,\sigma.$$

The factor $\frac{m}{OA^2}$ is the strength of field at A due to a pole m placed at O, and $\frac{m}{OA^2}\cos\epsilon$ is this same strength of field resolved along AB, the normal to the shell, or the number of lines of force per unit area passing through the shell-element: and $\frac{m}{OA^2}\cos\epsilon\,.\,\sigma$ is the absolute number of lines of force due to m which pass through the shell-element.

Now any distribution of magnetism may be represented as a distribution of magnetic poles, and the above proposition be applied. The work done against any system of poles will be found by simply adding the number of lines of force due to each element of the magnetic distribution intercepted by the shell-element which gives simply the total flux of the lines of force in the magnetic field across the element. If

this number be represented by n, the potential of the field on the shell-element becomes $\rho \,.\, AB \,.\, n$.

If the shell be a simple shell, of strength j, the potential of any magnetic distribution on the magnetic shell is equal to $j\Sigma n = jN$, where N is the whole number of lines of force due to the given system intercepted by the shell or more correctly embraced by its edge.

Giving the direction of the lines of magnetic force their proper signs (Art. 208) we see that in the figure the lines of force pass from the positive to the negative side of the shell, and conversely if they pass from the negative to the positive side the potential of the field on the shell would have been negative. Hence generally the potential of any magnetic system on a simple magnetic shell is measured by $\pm jN$ where j is the strength of the shell, N the number of lines of force due to the given system enclosed by its edge, the plus sign being used when the lines of force pass from the positive to the negative side of the shell. This potential when positive of course measures the work which would be done in bringing up the magnetic shell from an infinite distance to the given position in the field, and when negative the amount of work which would be done by the shell if allowed to pass by frictionless constraint from an infinite distance to the given position (see Art. 41).

The foregoing proposition might be treated as a particular case of this, since the solid angle there defined is clearly the number of lines of force from a unit pole which would be intercepted by the shell.

220. The following six Articles, 221—226, which may be treated as corollaries to propositions II. and III. proved above, are of great importance.

221. In estimating the potential of any magnetic shell on a point, we must remember that if some of the elementary cones cut the shell twice, the potentials of these elements will be equal and of opposite sign, and may therefore be neglected in the general summation. It is evident, on

inspecting the figure, that in this case we have only to take account of the free edge.

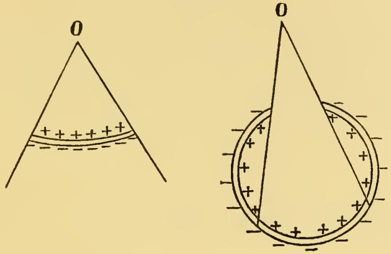

Fig. 76.

222. The potential of a closed shell on any internal point equals $\pm 4\pi j$, and on any external point vanishes.

The sum of all the solid angles subtended by elements of the shell at any internal point will clearly be the whole sphere, and this solid angle is measured by 4π, hence the potential is $\pm 4\pi j$. For any external point we notice that *all* the elementary cones cut the shell twice, and the whole potential therefore vanishes.

223. In the case of a shell in the form of a plane lamina the potential anywhere on the positive side is $2\pi j$, and on the negative side $-2\pi j$, for the solid angle subtended by a plane at any point on the plane is clearly a hemisphere. We see also that the work done in bringing a unit pole from the negative round to the positive side of the shell is $4\pi j$, and is independent of the path taken.

224. The potential of the plane lamina at any point in the plane of the shell outside the shell is zero. The solid angle subtended by the shell clearly in this case vanishes.

225. Since the potential measures the work done on a unit pole, we see generally that if a magnetic pole of strength m be moved in the field of a given shell from a position in which the solid angle subtended is Ω_1 to a position in which it becomes Ω_2 the work done will be measured by

$$mj(\Omega_2 - \Omega_1).$$

If Ω_2 be less than Ω_1 the work done is negative, or the pole can do work during the movement.

More generally if a magnetic shell be moved about in the magnetic field from a position in which the number of lines of force enclosed is N_1 to another in which the number of lines becomes N_2, the work done in the movement is

$$j(N_2 - N_1);$$

here also the work done is negative if $N_2 < N_1$, or the force acting on the shell *helps* the motion.

226. It appears from the last result that a magnetic shell free to move about in a magnetic field will place itself in a position where its potential is as low as possible, or in the position which includes the greatest number of negative lines of force.

227. Prop. IV. To find the strength of field, resolved perpendicular to its plane, of any uniform thin plate of attracting matter.

Let AB be the trace on the plane of the paper of the plate, and P a point in it round which an element ab of the plate is taken. Let O be the attracted particle (at which we assume unit mass), ON, the perpendicular on the plane of the plate, and let OPN be the plane of the paper. Let $a'b'$ be the projection of ab on a plane perpendicular to OP.

The attraction of the element ab on O

$$= \frac{\rho\sigma}{OP^2},$$

where σ is the area of ab and ρ the density.

Hence component along ON

$$= \frac{\rho\sigma \cos PON}{OP^2}.$$

But since ON is perpendicular to ab, and OP to $a'b'$, the angle $PON =$ the angle between ab and $a'b'$. Hence

$$\sigma \cos PON = \sigma', \text{ the area of } a'b'.$$

Hence component along ON

$$=\rho\cdot\frac{\sigma'}{OP^2}.$$

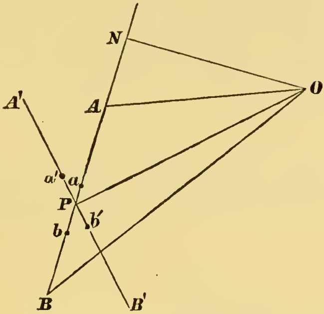

Fig. 77.

But (Art. 218) $\dfrac{\sigma'}{OP^2}$ is the area cut off a unit sphere whose centre is O by a cone which has O for its vertex and ab for its base, or the spherical measure of the solid angle subtended by ab. If this be denoted by ω the attraction along ON of the element ab is $\rho\omega$.

The same will be true for each element of AB, and we shall have for the whole attraction along ON,

$$\Sigma\rho\omega=\rho\Sigma\omega, \text{ since the plate is uniform,}$$
$$=\rho\Omega,$$

where Ω is the spherical measure of the solid angle subtended by AB. It is obvious that Art. 36 is only a particular case of this general proposition.

Cor. This proposition can be at once applied to every cylindrical bar magnet, since it consists only of a distribution of magnetism of density $+\rho$ over one end and $-\rho$ over the other. Hence if Ω_1, Ω_2 be the solid angles subtended at any point by its ends, the strength of the magnetic field parallel to its length will be $\rho\,(\Omega_1 \pm \Omega_2)$, *plus* if the attracted particle lie between the planes formed by the ends produced and *minus* if outside these planes

228. *Experiment* 7. A small closed voltaic circuit placed in the magnetic field is acted on by forces proportional to the forces which would be experienced by a thin magnetic shell whose edge coincides with the circuit, the strength of the current bearing a certain proportion to the strength of the shell, the direction of the current being such that an observer looking down on the north side of the shell sees the current following a direction opposite to the hands of a watch.

229. This experimental law can be extended to a voltaic circuit of any size and shape whatever. For conceive the voltaic circuit filled up by a surface, and this

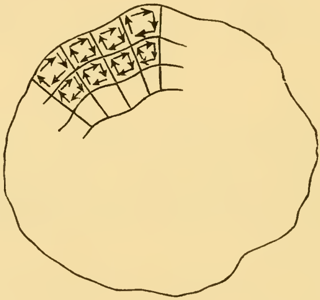

Fig. 78.

surface divided into a number of closed curves by lines cutting each other at right angles, whose distances are small compared with the curvature of the surface. Conceive currents of the same strength to circulate round each of these closed curves, as shown in the figure, all in the same direction.

Each closed curve may be regarded as a plane circuit, and for it by the above experiment may be substituted a small magnetic shell whose strength is in a certain ratio to the current-strength, and similarly for all the other elementary circuits; and the magnetic shell-elements substituted for each will all have the same strength.

But these shell-elements will make up a simple magnetic shell whose edge coincides with the original closed circuit.

Again, in the figure it is evident that along each side of the elementary closed circuit will be two currents of equal strength in opposite directions, which will therefore neutralize each other; the only parts not neutralized in this way being the elements which compose the original voltaic circuit.

Hence we see that as far as actions in the magnetic field are concerned we may substitute for any voltaic circuit a magnetic shell whose edge coincides with the circuit carrying the current, and whose strength bears a certain ratio to the current strength.

230. We may define the positive direction of the current in the circuit in the following way:

DEF. DIRECTION OF CURRENT. *The positive direction of the current is related to the positive direction of the lines of force in the same way as the direction of rotation to that of propulsion in a right-handed screw.*

This direction can be conveniently remembered by the twist in the muscles of the wrist in driving in a corkscrew. The opposite direction will be referred to as a left-handed screw, and the set of directions indicated above will be referred to as *right-* and *left-handed cyclical order*.

231. The experiment and deductions given above form the basis of the science of Electro-Magnetism.

It is usual here so to change our fundamental units that the current traversing a circuit and its equivalent magnetic shell shall have the same number to express their strength. Our former units of electromotive force, resistance, &c. will all have to be altered; but we shall assume at present that they are altered in such proportion that Ohm's formula remains unchanged, as also the formula for energy expended in the circuit.

The relations of the various units in the electrostatic and electromagnetic systems to each other we shall indicate in the next chapter.

232. We have now shown that the forces acting on a magnetic shell and on a voltaic circuit coinciding with its edge are identical, and since these forces are, in the case of the shell, derived from a magnetic potential, we shall assume that an identical electromagnetic potential exists from which the forces acting on the voltaic circuit are derived.

Before doing so, we must notice that the potential on a magnetic shell in a magnetic field is positive when the lines of force pass from the positive to the negative side of the shell (Art. 219), while the current in the equivalent circuit is left-handed or negative to these lines of force (Arts. 228 and 230). We must remember therefore in applying propositions proved for magnetic shells to voltaic circuits, that work done will be represented by potential with sign changed.

233. To keep this clearly before the student, we place here the properties proved for a magnetic shell, while in a parallel column we place the conventions made and properties deduced for a voltaic circuit.

MAGNETISM.

Prop. I. The potential on a magnetic shell in a magnetic field is measured by the product of its strength into the number of lines of magnetic force, counted algebraically, which it encloses. Art. 219.

Prop. II. The numerical value of the potential at a point, due to a magnetic shell, is equal to the strength of the shell multiplied by the solid angle subtended at the point by its edge. The + sign being attached when from the given point you look on the positive, or North, side of the shell. Art. 218.

ELECTRO-MAGNETISM.

Prop. I. The potential on a voltaic circuit in a magnetic field is measured by the product with sign changed of the current-strength into the number of lines of magnetic force counted algebraically enclosed by the circuit.

Prop. II. The numerical value of the potential at a point, due to a voltaic circuit, is equal to the product of the current-strength multiplied by the solid angle subtended at the point by the circuit. The + sign is attached when on looking down on the circuit from the point the current appears to follow the direction of the hands of a watch.

Magnetism.

Prop. III. The potential of a plane magnetic shell of strength j is on one side $2\pi j$ and on the opposite $-2\pi j$, the difference $4\pi j$ representing the work done in bringing a unit pole round the edge from the negative to the positive side. Art. 223.

Electro-Magnetism.

Prop. III. The potential of a plane voltaic circuit, carrying a current of strength i, is on one side $2\pi i$ and on the other $-2\pi i$, the difference $4\pi i$ representing the work done on a unit pole in bringing it round *outside the circuit* from the negative to the positive side.

Cor. Since in the case of a voltaic circuit the work done in passing just through the plane of the circuit must be zero, we conclude that the potential of this plane must be $\pm 2i\pi$, and that in measuring the difference of potential for all other places we must remember that it will be $4\pi i$, greater or less according as the path pursued passes through the circuit or round outside it.

Magnetism.

Prop. IV. The work done on a unit pole in carrying it from a point A where the potential of a shell is $j\Omega_1$ to a point B where the potential is $j\Omega_2$ is

$$j(\Omega_2 - \Omega_1).$$

Art. 225.

Electro-Magnetism.

Prop. IV. The work done on a unit pole in bringing it from a point A at which the potential of a voltaic circuit is $i\Omega_1$ to a point B at which the potential is $i\Omega_2$ is

$$-i(\Omega_2 - \Omega_1),$$

assuming the path pursued not to go through the circuit. If the path pass through the circuit it is represented by

$$i(\pm 4\pi - \Omega_2 + \Omega_1),$$

the $-$ or $+$ being taken according as the path through the circuit is positive or negative relatively to the lines of force.

MAGNETISM.	ELECTRO-MAGNETISM.
Prop. V. The work done on a shell placed in a magnetic field, and moved from a position in which N_1 lines of force intersect it to a position in which N_2 lines of force intersect it, is $j(N_2 - N_1)$. Art. 225.	**Prop. V.** The work on a voltaic circuit placed in the magnetic field, and moved from a place in which N_1 lines of force intersect it to a position in which N_2 lines of force intersect it, is $-i(N_2 - N_1)$.
Prop. VI. If $N_2 < N_1$ the work done is negative, or the shell acquires kinetic energy owing to the magnetic forces helping the movement. Art. 225.	**Prop. VI.** If $N_2 > N_1$ the work done is negative, that is the circuit acquires kinetic energy owing to the electromagnetic forces assisting the movement.
Prop. VII. A magnetic shell capable of movement in the magnetic field places itself so as to include the smallest possible number of lines of force, or, what is the same thing, the greatest possible number of negative lines of force. Art. 226.	**Prop. VII.** A voltaic circuit free to move places itself in the field so as to include the greatest possible number of lines of force. That is, it will place itself in the strongest part of the field in such a position that the lines of force are as nearly as may be perpendicular to it, the current being related to the direction of the lines of force in right-handed cyclical order.

234. Since the potential at a point depends on the solid angle subtended by the circuit, we see that the surfaces over which the potential is constant will emanate from the circuit and will form bowl-shaped surfaces having the circuit for their edge. A system of equipotential surfaces would be a system of such unclosed surfaces intersecting each other at finite angles in the given circuit.

In assigning numerical values to the surfaces, we must remember that the potential represents the work done in carrying a unit pole from the surface to an infinite distance, and this depends on whether the path pursued

passes through the circuit or round its edge, and if it passes through the circuit, on how many times it passes through the circuit always in the same direction. Hence we cannot assign a fixed value to a given equipotential surface, but a series of values differing by $4\pi i$. The work done on a unit pole in bringing it from a surface whose potential is V_1 to another whose potential is V_2 is

$$\pm 4n\pi i + V_1 - V_2,$$

where n is the number of times the path pursued passes through the circuit in the same direction; if that direction be with the lines of force we prefix the $-$ sign, and if against them the $+$ sign.

The path of the pole which in the last paragraph passed through the circuit n times without returning, may be said conveniently to be *linked* n times with the circuit.

The lines of force cut all equipotential surfaces at right angles, and are therefore a system of oval curves with the conducting wire passing through them. In conformity with the convention just made we may say that the lines of force are linked with the circuit, and the circuit with any one of its lines of force may be conceived as two successive links in a common chain.

235. We have already stated that the current is related to the lines of force in right-handed cyclical order. If we now conceive the line of force as a closed curve and the circuit as a direction cutting through it, the positive direction of the line of force will be related to that of the current in right-handed cyclical order. Hence both the lines of force are related to the current and the current to the lines of force in right-handed cyclical order.

This rule is clearly equivalent to that usually given, that a figure swimming in the current which enters by its heels and leaves by its head, and looking towards the magnet, sees the north pole driven to its left.

If the pole be fixed, and the current free to move, it is clear that the current will be driven round a north pole, so that the figure in the current looking towards the

pole is always moved towards its right hand. A convenient rule for remembering this direction, often useful in practice, is that *a figure swimming in the current, looking along the lines of force, will, with the conductor, be carried towards its left.*

236. Prop. VIII. In computing the potential on any closed circuit in a magnetic field we may substitute for it any closed circuit which is obtained by projecting the given circuit by means of lines of force.

For since lines of force never intersect except at a magnetic pole we cannot by this means alter the number of lines of force enclosed.

Cor. 1. In any movement of a conductor the change in potential produced by the movement of any part of the closed circuit parallel to lines of force, or parallel to planes containing the lines of force in that part of the field, is nil.

Cor. 2. For any sinuous conductor a straight one may be substituted.

It is clear that a straight line can always be drawn through any sinuous line such that the number of lines of force omitted may be just counterbalanced by the number of extra lines of force included on substituting the straight for the sinuous current.

237. Prop. IX. If two circuits more or less parallel to each other carry currents in the same direction they attract each other, and if the currents be in opposite directions they repel each other.

Let A be a portion of a conductor carrying a current, and let the plane of the rest of the circuit be more or less perpendicular to the paper. Then it is clear (Art. 236) that the lines of force are a system of oval curves, rising from the paper to the left of A, and sinking into it to the right of A. If B be a parallel conductor carrying a current in the same direction, the lines of force enclosed by B, and on which the potential of B depends, will be all those which

fall to the right of B, and remembering the rule we see that B's current is positive to these lines of force.

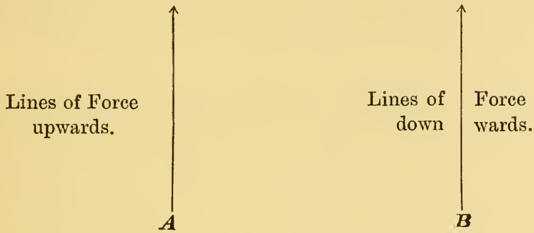

Fig. 79.

The electromagnetic action in the field will therefore (Art. 233, Prop. VII.) tend to place B so as to enclose more lines of force, that is, will draw B towards A. If the current in B be opposite to that in A, B's current will be negative to the lines of force, and the electromagnetic force will be therefore repulsive.

Cor. 1. If there be two straight wires parallel to each other carrying currents they will, if the currents be in the same direction, attract, and if in opposite directions, repel each other.

Cor. 2. If the two wires in the last corollary be inclined to each other and the currents both run into or out of the corner made by the wires, they will attract each other, but if one run into, and the other out of the corner, they will repel each other.

Let XOX' and BAB' be the two conductors, OA being

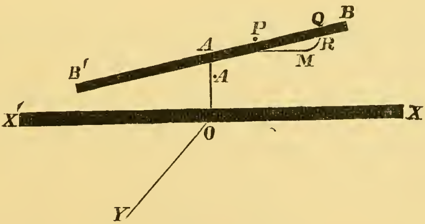

Fig. 80.

perpendicular to both. Let OY be at right angles to OX in a plane parallel to BAB'.

The lines of force due to XOX' will clearly be (neglecting its ends) a system of circles in planes perpendicular to it.

Let PQ be an element of BAB', and let it be projected into the bent line $PMRQ$ in which PM is parallel to OX, MR to OY, and RQ to OA. The parts QR, RM will be parallel to the plane containing the lines of force. Hence moving the conductor parallel to itself, we have only to consider the change in potential due to the movement of PM. If the currents in XOX' and BAB' both run into or out of the corner, PM and OX will be parallel currents in the same direction, and will attract each other. If one run into and the other out of the corner, the currents in PM and OX will be opposite, and they will therefore repel each other.

238. Prop. X. If we have two voltaic circuits A and B the number of A's lines of force which B encloses will be equal to the number of B's lines of force which A encloses, when the current-strength in each is unity.

This proposition might have been at once inferred from the general principle of mutual potential in any two systems (Art. 41).

Let N_1 be the number of lines of force due to A enclosed by B, and N_2 the number due to B enclosed by A, and let i_1 and i_2 be the current-strengths in A, B respectively.

The work done in carrying B out of A's field of force will then be $N_1 i_2$.

The work done in carrying A so as to hold the same position in space relative to B will be $-N_2 i_1$.

Hence we see that an amount of work $N_1 i_2 - N_2 i_1$ would be expended in carrying a magnetic system against no external magnetic forces from one place to another, and this must vanish. Hence

$$N_1 i_2 - N_2 i_1 = 0,$$

or

$$N_1 i_2 = N_2 i_1.$$

But since potential and consequently force at any point in the field of a voltaic circuit is, by Prop. II., proportional to current-strength, we may make $N_1 = M_1 i_1$, and $N_2 = M_2 i_2$; we see then that

$$M_1 i_1 i_2 = M_2 i_1 i_2,$$

or

$$M_1 = M_2.$$

But M_1 and M_2 will be the number of lines of force enclosed respectively by A and B when there is unit current in each. Hence the proposition.

DEF. COEFFICIENT OF MUTUAL INDUCTION. *The quantity* M *in the preceding proposition which gives the number of lines of force due to one of two circuits (each carrying unit current) enclosed by the other is defined to be the coefficient of mutual induction between them.*

If M be the coefficient of mutual induction between two circuits carrying currents i_1 and i_2 the potential of each due to the other's magnetic field will be $Mi_1 i_2$. If the two circuits be free to move in the field, they will clearly place themselves in such a position that M shall be as large as possible; this will be when the two circuits are as nearly as possible in the same plane and carry parallel currents.

239. Prop. XI. To find an expression for the whole energy in a circuit carrying a current.

That a voltaic current is a source of energy we have already seen, and when the circuit is separated from all other circuits its energy must clearly be kinetic. Whether this energy be that of moving electricity or of the movement of the conductor carrying the electricity, or of both combined, we cannot here enquire. In either case the analogy of the vis viva or kinetic energy of moving material bodies would lead us to conclude that it depends on the square of the current-strength. We see also that the potential energy of two circuits carrying currents depends on the geometry of the circuits and on the product of their current-strengths, and since all forms of energy must be of the same order, we may infer that the energy of a given circuit will be represented by a certain coefficient depending on the geometry of the circuit multiplied by the square of the current-strength.

In a circuit let us call the coefficient L, and the current-strength i, the energy of the circuit will then be Li^2.

Place another conductor in all respects similar to the former, so as to coincide with it, and let it carry the same current in the same direction.

The energy of this system will clearly be represented by $L(2i)^2 = 4Li^2$, since the geometry is unaltered and the current doubled.

If the conductors be separated so as to be carried out of each other's field, their whole energy is reduced to the sum of their separate energies, or $2Li^2$.

Hence the work done in separating them is the difference, or $2Li^2$.

But by the previous Proposition, the work done in separating them is equal to the number of lines of force due to one enclosed by the other. When the circuits coincide each one encloses *all* the lines of force due to the other.

Let the quantity denoted by M, when the circuits coincide, be represented by M_0. This symbol then represents the whole number of lines of force embraced by the circuit carrying unit current. Hence the work done in separation is $M_0 i^2$ when each carries current i:

$$\therefore\ 2Li^2 = M_0 i^2,$$
$$L = \tfrac{1}{2}M_0.$$

It is now more usual to make M_0 the coefficient of self-induction, still however retaining for it the letter L. In this case the intrinsic energy of the circuit is given by $\frac{1}{2}Li^2$.

DEF. COEFFICIENT OF SELF-INDUCTION. L *is defined to be the coefficient of self-induction of a circuit, and is equal to the number of lines of force embraced by the circuit, when removed from all other circuits and carrying unit current.*

COR. If there be two circuits carrying currents, and if L, N be their coefficients of self-induction, and M the coefficient of mutual induction, the whole energy of the field when the current-strengths are i_1, i_2 is given by the expression

$$\tfrac{1}{2}Li_1^2 \pm Mi_1 i_2 + \tfrac{1}{2}Ni_2^2,$$

the positive or negative sign being given to the middle term as the lines of force from one circuit pass in the positive or negative direction through the other.

240. Having obtained the foregoing expression for the energy of a voltaic circuit carrying a current, we get the idea of inertia to be overcome in establishing the current at first, or making any alteration in it when once established. On applying an electromotive force to a circuit, part of the energy of the battery is used up in overcoming the resistance or in heating the circuit, while the remainder goes to increase its kinetic energy or to do work external to the circuit.

This can be expressed by equating the energy taken from the battery in any given short interval to the sum of the heat developed, the increase in kinetic and potential energy and the external work.

If E be the electromotive force of battery, and i the current-strength, the energy subtracted in a given time $\tau = Ei\tau$.

If R be the external resistance the energy expended in heat $= Ri^2\tau$.

If L be the coefficient of self-induction and if i' be the current-strength at the end of a time τ, the rise of intrinsic energy $= \frac{1}{2}L(i'^2 - i^2)$.

If during the time τ the circuit be moved by electromagnetic forces from a position in which it embraces N lines of force of some external magnetic field to one in which it embraces a larger number N', the energy drawn from the battery to make this movement $= (N' - N)i$. If W be the rate of working of an external machine, working by the electromagnetic forces in the circuit, the energy required to keep it going for time $\tau = W\tau$.

Hence equating the energy given out from the battery to the sum of all the energy used up gives

$$Ei\tau = Ri^2\tau + \tfrac{1}{2}L(i'^2 - i^2) + i(N' - N) + W\tau.$$

Remembering that $i'^2 - i^2 = (i' - i)(i' + i) = 2i(i' - i)$ very

nearly when $i'-i$ is very small, this equation reduces at once on division by $i\tau$ to

$$E = Ri + L\frac{i'-i}{\tau} + \frac{N'-N}{\tau} + \frac{W}{i}.$$

241. Prop. XII. To calculate the law of establishment of the current in a conductor when an electromotive force is applied to it.

This will be a particular case of the preceding article where the external work vanishes. Hence we have

$$\begin{aligned} Ei\tau &= Ri^2\tau + \tfrac{1}{2}L(i'^2 - i^2) \\ &= Ri^2\tau + Li(i'-i); \end{aligned}$$

$$\therefore\ E\tau = Ri\tau + L(i'-i) \qquad \text{(1)},$$

which may be written

$$\left(\frac{E}{R} - i\right)\tau = \frac{L}{R}(i'-i).$$

Let $\dfrac{E}{R} - i = y$, and $\dfrac{E}{R} - i' = y'$;

$$\therefore\ i' - i = -(y'-y);$$

$$\therefore\ y\tau = -\frac{L}{R}(y'-y);$$

$$\begin{aligned} \therefore\ \tau &= -\frac{L}{R}\frac{y'-y}{y} = -\frac{L}{R}\log\left(1 + \frac{y'-y}{y}\right) \\ &= -\frac{L}{R}(\log y' - \log y), \end{aligned}$$

which is a form suitable for direct summation. Remembering that $y = \dfrac{E}{R}$ when $i = 0$, we have after time t,

$$t = -\frac{L}{R}\left\{\log\left(\frac{E}{R} - i\right) - \log\frac{E}{R}\right\};$$

$$\text{or}\quad \frac{\dfrac{E}{R} - i}{\dfrac{E}{R}} = \epsilon^{-\frac{R}{L}.t};$$

$$\therefore\ i = \frac{E}{R}\left(1 - \epsilon^{-\frac{R}{L}.t}\right).$$

The ratio $\frac{R}{L}$ is generally large and the rise in strength of the current takes place with such rapidity that we cannot observe its rise. In marine telegraphy, where L, the coefficient of self-induction, is large and complicated by the Leyden-jar action of the insulating sheath, the term $\epsilon^{-\frac{R}{L}.t}$ leads practically to a lengthening out of the signal, so that a sharp signal transmitted to the wire by closing the circuit for an instant shows in a galvanometer at the other end a gradual rising and falling again of the current.

242. We may compare the establishment of a steady current in a conductor to the establishment of steady motion in a steam-engine. When the locomotive is moving along steadily the whole work done by the steam on the piston is used up in friction on the rails and in the machine. When the engine is quickening its pace the work done by the steam is greater than that used up in friction, and the difference goes to increase the kinetic energy of the system, and vice versa when the engine is pulling up.

So with electricity in motion. When steady, Ohm's law expresses the fact that the energy given out by the battery is converted into heat in the circuit. When however the current is increasing the energy given out by the battery is more than that used up in the circuit, and the remainder goes to increase its kinetic energy, and vice versa when the current is ceasing. In practice the current becomes steady so rapidly that we only observe the indirect effect of the increasing energy in the extra spark.

243. We have seen that every voltaic circuit possesses an electromagnetic field, and in this field exerts attractions and repulsions upon magnetic poles or other voltaic circuits. If in obedience to these attractions and repulsions movements take place, the law of conservation of energy shows us that the work done by the circuit must be done in some way at the expense of the energy in the circuit. This energy we have just seen to be kinetic, depending on the geometry of the circuit and the current-strength.

The only way therefore in which energy can be abstracted from or added to the circuit is by a diminution or increase of current-strength. The diminution of current-strength will last just long enough to compensate for the work done, and the steady current will be established again. These variations of current-strength in the circuit while work is being done in the electromagnetic field are known as induced currents. Since the induced current is always a compensation for the energy expended or gained in the field, it is clear that acting alone it would oppose the movement, or its direction will always be such that by its electromagnetic effect it would oppose the movement taking place in the field. This is known as Lenz's Law.

From Lenz's law combined with the law stated in Art. 235, we have the rule for the direction of an induced current in a moving conductor, that *a figure in the conductor looking along the lines of force and moved towards his left with the conductor will experience an induced current which enters by his head and leaves by his heels.*

244. Prop. XIII. To calculate the induced current produced by the movement of any conductor in a magnetic field.

In this case no external work is done and the term containing W in the equation of Art. 240 may be omitted. Moreover the term containing the self-induction L will be negligible in all cases where we obtain an approximately steady current, or a current which changes slowly, as well as in cases where L is very small compared with R and N, which will be the case if the moving coil has no iron core.

With these approximations we have

$$Ei\tau = Ri^2\tau + (N' - N)\,i,$$

or

$$E\tau = Ri\tau + (N' - N).$$

Let i_0 denote the steady current, then $E = Ri_0$. Hence

$$R(i - i_0)\,\tau = -(N' - N).$$

But $i - i_0$ is the induced current-strength and $(i - i_0)\,\tau$ is the quantity of electricity transmitted during the time τ. Adding for all the short intervals of the movement,

$$\Sigma\,(i - i_0)\,\tau = -\frac{\Sigma\,(N' - N)}{R}.$$

But $\Sigma(i - i_0)\tau$ is the quantity of electricity transmitted during the whole movement and may be denoted by $[i]$. This is often called the total induced current.

$\Sigma(N' - N)$ will be simply $N_1 - N_0$ when N_1 is the number of lines of force enclosed at the end, and N_0 the number at the beginning of the movement;

$$\therefore\ [i] = -\frac{N_1 - N_0}{R}$$

$$= -\frac{\text{Number of lines of force added}}{\text{Resistance of Circuit}}.$$

COR. 1. If the increase of lines of force takes place at a uniform rate the current will be constant and measured by $\frac{[i]}{t}$, where t is the time. Also since $E = Ri$ the E.M.F. is given by

$$-\frac{\text{Number of lines of force added}}{t},$$

or the number of lines added per second. This will be true however short the time t, and we learn that the E.M.F. of an induced current is always the rate of increase of the lines of magnetic force embraced by the circuit, with sign changed.

COR. 2. The expression for the E.M.F. and the induced current is independent of the E.M.F. initially in the circuit. Hence we see that we get induced currents by the movement of circuits in a magnetic field, though there is initially no E.M.F. in the moving circuit.

COR. 3. If a straight conductor forming part of a closed circuit is carried across lines of magnetic force, the electromotive force of the induced current is $-Hlv$, where H is the number of lines of force per unit area or the strength of the field perpendicular to the plane of the conductor's motion, l the length of the conductor, and v the velocity with which it moves parallel to itself.

Let AB be the conductor, and let the rest of the circuit be completed by thick bars A, C, B whose resistance may be neglected.

If the conductor move from AB to $A'B'$, and the lines of force be perpendicular to the paper, the number of lines of force added by the movement

$$= H \times \text{area } ABA'B' = H \times AA' \times l.$$

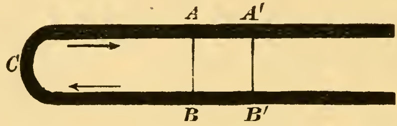

Fig. 81.

If $[i]$ denote the total current,

$$[i] = -\frac{H \times AA' \times l}{R}.$$

If the time occupied by the movement is t, the current

$$i = \frac{[i]}{t} = -\frac{H \times AA' \times l}{R \cdot t},$$

the direction of this current being as in figure, if the lines of force are upwards.

But if the conductor move uniformly from AB to $A'B'$ in time t, $\frac{AA'}{t} = v$, the velocity of the motion;

$$\therefore \; i = -\frac{Hvl}{R};$$

$$\therefore \; Ri = -Hvl,$$

and $\qquad Ri = E; \qquad \therefore \; E = -Hvl.$

We see the current will be strongest when the conductor is moved parallel to itself and perpendicular to the lines of force, the direction of the induced current being perpendicular to these two directions.

Cor. 4. We see also that there will be an induced current during the opening or closing of the circuit.

1st. *At closing the circuit.* Since L, the coefficient of induction, is the number of lines of force enclosed by a

circuit carrying unit current, the whole number added will be Li_0, where i_0 is the steady current;

$$\therefore\ [i] = -\frac{L}{R}\, i_0.$$

2nd. *At opening the circuit.* The number of lines *subtracted* will be Li_0, hence

$$[i] = +\frac{L}{R}\, i_0.$$

Since $$i_0 = \frac{E}{R}, \qquad [i] = \frac{EL}{R^2},$$

and for the average electromotive force in either case

$$E' = R \times \frac{[i]}{t} = \frac{EL}{Rt},$$

where t is the time the current lasts.

This result is interesting, since we see that by reason of the extreme smallness of t, E' may be many multiples of E. Thus the induced current having very high electromotive force is able to break across a finite air-space, giving rise to the galvanic spark or extra current, although no cell can break directly across an appreciable air-space. Sir W. Thomson says, that 5510 Daniell's cells would be required to give a spark between two brass terminals about $\frac{1}{8}$ in. apart.

245. Extending the analogy pointed out in Art. 242 we may compare induced currents with the phenomena attending the establishment of steady motion in a locomotive and train. At the first start the engine has to overcome the inertia of the train behind it, and as a consequence receives a number of impulsive jerks backwards just analogous to the induced negative current at closing the circuit. When the engine slackens pace it has to overcome the energy of the moving mass behind, and accordingly receives jerks forwards analogous to the induced positive current at opening the circuit.

246. Prop. XIV. The rate of working of any electromagnetic engine will be the greatest possible when starting the engine diminishes the current in the circuit by one half.

Let W be the rate of working of the engine, i the current when the engine is stopped, and i' the current when the engine is working. Then Art. 240,

$$Ei\tau = Ri^2\tau \text{ when the engine is at rest.}$$

$$Ei'\tau = Ri'^2\tau + W\tau \text{ when the engine is at work.}$$

Whence by division

$$Ri'^2 + W = Rii',$$

$$W = R(ii' - i'^2),$$

$$= R\left\{\frac{i^2}{4} - \left(\frac{i}{2} - i'\right)^2\right\}.$$

The right-hand side is a maximum when

$$i' = \frac{i}{2},$$

which proves the proposition.

Examples on Chapter VIII.

1. Two magnetic compasses are placed on a table near each other; explain how they influence each other's directions in all different positions.

2. A common bar magnet is placed on a table and a compass needle lies on the table subject to the magnet's force; show by a diagram the positions of the needle in different positions relatively to the magnet.

3. A dipping-needle is free to move in a plane at a given inclination to the magnetic meridian; show how to find the apparent dip.

4. Show that if the apparent dip observed in any two planes at right angles to each other be δ_1, δ_2, then δ the true dip can be found from the formula

$$\cot^2 \delta = \cot^2 \delta_1 + \cot^2 \delta_2.$$

5. If the dipping-needle move in a plane perpendicular to the meridian, show that it will remain vertical. Hence show how to determine the plane of the meridian by observations with the dipping-needle only.

6. If r, r' be the distances of a point from the north and south pole of a magnet respectively, show that for any point on a given equipotential surface $\frac{1}{r} - \frac{1}{r'}$ is of constant value.

7. Find the points in which any given equipotential surface cuts the magnetic axis.

Ans. If m be the strength of each pole and a the length of the axis, then the surface whose potential is $\frac{m}{c}$ will cut the axis at a distance from the north pole $\frac{1}{2}\{a + 2c - \sqrt{a^2 + 4c^2}\}$, and the axis produced at a distance $\frac{1}{2}\{\sqrt{a^2 + 4ac} - a\}$.

8. If in the preceding question ψ, ψ' be the angles which r, r' make with the tangent plane to an equipotential surface, $\frac{\cos\psi}{\cos\psi'} = \frac{r^2}{r'^2}$.

9. If ϕ, ϕ' be the angles which r, r' in the preceding question make with a line of force, $\frac{\sin\phi}{\sin\phi'} = \frac{r^2}{r'^2}$.

10. If θ, θ' be the angles which r, r' make with the magnetic axis produced in one direction, show that along any line of force, $\cos\theta - \cos\theta'$ is constant.

11. Let two equal rods Nn, Ss turn on pivots about points N, S which are the poles of a given magnet. Then if they be moved so that ns is always perpendicular to NS, or NS produced, the intersection of Nn, Ss will trace out a magnetic curve. (See Roget's *Electricity.*)

12. Prove the following construction for obtaining any number of points on a system of magnetic curves:—Divide the magnetic axis into any integral number of parts, and set off along the axis produced any large number of equal parts. With centres N, S describe two equal circles having

for any radii as large multiples of this subdivision as practicable, and divide their circumferences by perpendiculars drawn to the axis at each subdivision: draw lines joining N and S to the points of division of these circumferences; the lines of force will then be curvilinear diagonals of the lozenge-shaped spaces into which the figure is divided. (Roget's *Electricity*.)

13. Show that the equipotential surfaces will be closed and the lines of force constantly divergent from either pole when the magnetic system consists of two similar and equal poles at a distance from each other.

14. Show that the lines of force for two similar poles will be the other set of curvilinear diagonals of the lozenge-shaped spaces indicated in ques. 12.

15. Find the form of the surface of zero potential for any bar magnet, and show that the resultant magnetic force for points on it is given by $\frac{ml}{r^3}$, where m is the strength of each pole, l the length, and r the distance of the point from one pole.

16. A closed voltaic circuit is supported at its centre of gravity but otherwise free. Explain the position assumed by it under the action of the earth's magnetism.

17. A straight conductor (capable of sliding freely on fixed bars and forming with them a closed voltaic circuit) carries a current. Explain the direction of its movement,

(i) When capable of moving parallel to itself in the horizontal plane and carrying a current from North to South.

(ii) When capable of moving parallel to itself in the horizontal plane and carrying a current from East to West.

(iii) When capable of moving parallel to itself in the vertical plane and carrying a current from North to South.

(iv) When capable of moving parallel to itself in the vertical plane and carrying a current from East to West.

18. A straight conductor carrying a current is capable of rotation round a magnetic pole. Show in all cases the relation between the sign of the pole, the direction of the current, and the direction of rotation.

19. Discuss the previous question in the case when the conductor is at rest and the magnetic pole free to move.

20. Show that a straight horizontal conductor placed East to West and carrying a current will, if exactly balanced, appear to lose or gain weight when the direction of the current is reversed. At what part of the earth will this effect be strongest?

21. A straight conductor carrying a current is fixed at one end, and the other rests by help of a cork-float in contact with mercury. Show that the conductor placed anywhere on the earth's surface will rotate, but that except at the magnetic poles of the earth the rate of rotation will vary in different parts of its course.

22. A long wire carrying a current has a short straight wire also carrying a current perpendicular to it but not crossing it. Investigate the direction of movement (if any) in the short conductor for different directions of the currents.

23. Investigate the direction of rotation in Barlow's wheel for given directions of the lines of magnetic force and of the current.

24. A bar of soft iron has a coil of wire round it. Show by a diagram the direction of the current induced in the coil, (i) when a N. pole approaches one end of the bar, (ii) when the same pole is removed.

25. A bar magnet is drawn completely through a hollow coil of wire forming a closed circuit. Show the direction, and roughly the variations in strength, of the induced current during the movement.

26. A straight wire forming part of a closed conductor is placed horizontally and slides parallel to itself, (i) from E. to W., (ii) from N. to S. Show in each case the direction of the current induced in it.

27. The same wire is arranged so that it can move in a vertical plane. Find the direction of the induced current, (i) when it rises vertically in a plane perpendicular to the meridian, (ii) when in the plane of the meridian.

28. Barlow's wheel (see ques. 23) has the battery removed but the battery circuit closed and the wheel made to rotate by mechanical means. Find the direction of the induced current.

29. A wire in the form of a closed circle rotates about a vertical axis in its own plane in the direction of the hands of a watch; investigate the direction of the induced current for different positions. Show that a small magnet suspended at the centre of the rotating coil will have its N. pole deflected towards the East if the rotation of the coil be with the hands of a watch. (See *B. A. Reports on Electrical Standards.*)

30. A wire parallelogram carrying a current is suspended from Ampère's stand, and allowed to take up its position of rest under the action of the earth alone. Explain its position of rest.

31. Show that a wire bent into the form of the figure 8 and carrying a current, will be astatic in relation to the earth.

32. A magnet is suspended horizontally over a diameter of a rotating copper disc. Show that the magnet will on rotating the disk be deflected from the meridian in the direction of rotation of the disk.

33. Explain the effect of a copper box on the oscillations of a magnet needle suspended within it.

34. A copper strip is drawn between the poles of a powerful horse-shoe magnet, and its opposite edges are connected by springs with galvanometer terminals. Show the direction of the induced current.

35. Show that the induced currents in the copper strip will retard the movement across the field.

36. Explain the difficulty of drawing a metal sheet between the poles of a powerful electromagnet.

37. A wire parallelogram is suspended on an Ampère's stand, and a circular copper plate rotated below it. Show the direction of the surface currents in the plate, and show that the circuit will be deflected, following the direction of rotation of the plate.

38. If the conductor in the last question consist of two equal parallelograms with the current flowing as in a figure 8, the system will rotate following the rotation of the plate.

39. A metal band of a circular form is made to turn on a vertical axis through its centre and perpendicular to its plane. Two points above each other on the upper and lower edge are connected by springs with the terminals of a galvanometer.

(i) A wire carrying a current is placed vertically near the springs; show that there will be an induced current, and investigate its direction.

(ii) The wire is bent into a circle and placed so as to surround the upper edge of the band; investigate the direction of the induced current.

(iii) Will there be a current when the bent wire encircles the band near its middle?

(iv) Extend this to the case of a conducting sphere which turns about a diameter and has the bent wire placed so as to embrace an equatorial plane. Show that there will be *superficial* currents from the poles to the equator when the current and rotation are in the same direction.

(v) Show also the direction of the superficial currents when the bent wire is in a meridian of the revolving sphere.

CHAPTER IX.

ABSOLUTE DIMENSIONS OF PHYSICAL UNITS.

247. We stated in our first paragraph that all physical units are referable to the fundamental ideas of *space, time,* and *mass*, the units of which are arbitrary. These units once fixed, each definition we employ of a new unit contains implicitly its reference back to the absolute system. It is our object in this chapter to trace the measures of the units we have employed, and represent them in terms of arbitrarily assumed fundamental units.

248. We must remember that if we make any change in our unit the change produced in the measure varies inversely as the change in the unit. Thus changing the unit of length from a foot to a yard, the measures of all distances will be divided by three, and the same principle applies in all cases.

249. There are two classes of units we have concerned ourselves with, *mechanical* units and *electrical* units, many of the latter having been referred to under two systems of measurement, the *electrostatic* and *electromagnetic*. We shall therefore divide our investigation into these three divisions, referring each time to the definition and the algebraical expression of it always implied.

We call the ratios of the new to the old units of *length, time,* and *mass* respectively L, T, M.

250. (1) *Mechanical Units.*

Velocity is defined (Art. 2) as space described per unit of time, and may therefore be measured by $\frac{\text{space}}{\text{time}}$. Hence, retaining our former symbols as far as possible,

$$v = \frac{L}{T} = LT^{-1} \qquad (1).$$

The meaning of this expression is, that given any change in the fundamental units (space and time), the change in the derived unit (velocity) is at once found by substituting in this formula the ratio between the new and old absolute units.

Acceleration is defined (Art. 4) as velocity added per second, and may be measured by $\frac{\text{velocity}}{\text{time}}$.

$$f = \frac{v}{T} = LT^{-2} \qquad (2).$$

Density is defined (Art. 7) as Mass per unit volume. Hence its dimensions are $\frac{\text{mass}}{\text{volume}}$ and the dimensions of volume are given by the cube of a length

$$D = \frac{M}{L^3} = ML^{-3} \qquad (3).$$

Specific gravity is defined as the ratio of two masses and hence is not altered by altering the system of units. It may be expressed by

$$s = M^0 L^0 T^0 = 1 \qquad (4).$$

Momentum is defined (Art. 8) as the product of mass and velocity, and is therefore measured by

$$Mv = MLT^{-1} \qquad (5).$$

Force is defined (Art. 13) as rate of change of momentum per second, and is measured by $\frac{\text{momentum}}{\text{time}}$.

$$F = \frac{MLT^{-1}}{T} = MLT^{-2} \qquad (6).$$

Energy may be defined either as work, that is (force) × (space), or as energy (Art. 24), that is $\frac{1}{2}$ (mass) × (velocity)^2. Denoting it by the general symbol W, we have in either case

$$W = ML^2T^{-2} \text{ .... (7).}$$

This will give the dimensions of every form of energy.

Rate of working is defined by energy per unit time and therefore is of the dimensions of $\dfrac{\text{energy}}{\text{time}}$

or $$ML^2T^{-3} \text{ .... (8).}$$

251. (2) *Electrostatic and Magnetic Units.*

Quantity. The measure of quantity depends ultimately on the law that the force between two equal quantities (Q) at a distance (L) from each other is measured by $\dfrac{Q^2}{L^2}$ (Art. 56).

Hence, as far as dimensions are concerned,

$$\frac{Q^2}{L^2} = F = MLT^{-2} \text{ .... (9);}$$

therefore $$Q = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}.$$

Magnetic pole, or *Quantity of Magnetism.* The same formula (Art. 208) expresses the force between two magnetic poles. Hence also

$$m = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \text{ ....(10).}$$

Electrical and Magnetic Density are defined (Arts. 57 and 207) as quantity of electricity or magnetism respectively per unit area, and are measured by $\dfrac{Q}{L^2}$ and $\dfrac{m}{L^2}$ respectively. Hence, in both cases,

$$D = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \text{ ....(11).}$$

Potential is defined (Arts. 60 and 208) as work done on a unit of electricity or magnetism, and is therefore measured by $\dfrac{W}{Q}$ or $\dfrac{W}{m}$ respectively. In both cases,

$$\left.\begin{matrix} V \\ E \end{matrix}\right\} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \text{ ....(12),}$$

since Electromotive Force is a form of Potential.

Electrical or Magnetic Force or Strength of Field is defined (Art. 60) as force on a plus unit, and may be measured by $\frac{F}{Q}$, which gives

$$\left.\begin{matrix}F\\H\end{matrix}\right\} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \qquad (13),$$

the symbol F being used in Electricity, for strength of field not of course with the same meaning as above in Mechanics.

Number of Lines of Force. Strength of Field is measured (Art. 209) by the number of lines of force per unit area. Hence number of lines of force is measured by HL^2, or

$$N = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \qquad (14).$$

Capacity is defined (Art. 70) as quantity per unit potential, and may be measured by $\frac{Q}{V}$. Therefore

$$C = L \qquad (15).$$

Specific Inductive Capacity is defined (Art. 77) by the ratio of two quantities of electricity, and is an abstract number. Hence in dimensions

$$K = M^0L^0T^0 = 1 \qquad (16).$$

Current-Strength is defined (Art. 158) as quantity per second, and is measured by $\frac{Q}{T}$; therefore

$$I = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \qquad (17).$$

Resistance is defined (Art. 162) by Ohm's law by the equation $E = RI$. Hence

$$R = \frac{E}{I} = \frac{T}{L} = L^{-1}T \qquad (18).$$

Conductivity is (Art. 161) the reciprocal of resistance, and is therefore measured by

$$LT^{-1} \qquad (19).$$

Specific Conductivity is defined (Art. 157) by the formula $I = cF\sigma$.

$$c = \frac{\text{current-strength}}{(\text{force}) \times (\text{area})} = \frac{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}} = T^{-1} \text{......(20)}.$$

Specific Resistance is defined (Art. 161) as the reciprocal of specific conductivity, and is measured by

$$\rho = T \text{ .... (21)}.$$

252. (3) *Electromagnetic Units.*

We shall denote electromagnetic measure by the same symbols as those employed in electrostatic, placing a bar over the symbol to indicate that it refers to this measurement.

Current-Strength. This is defined as being identical with the strength of a magnetic shell (Art. 231), which again is defined (Art. 218) as the product of magnetic density into length. Hence

$$\bar{I} = DL = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \text{..................(22)}.$$

Quantity is defined by the formula $\bar{I} = \frac{\bar{Q}}{T}$. Therefore

$$\bar{Q} = \bar{I}\,.\,T = M^{\frac{1}{2}}L^{\frac{1}{2}} \text{..................(23)}.$$

Potential or Electromotive Force is defined (Arts. 60 and 208) as the work done on unit quantity, and is therefore measured by $\frac{W}{\bar{Q}}$. Therefore

$$\left.\begin{matrix}\bar{V}\\ \bar{E}\end{matrix}\right\} = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \text{ .... (24)}.$$

Resistance is defined (Art. 162) by Ohm's law. Therefore

$$\bar{R} = \frac{\bar{E}}{\bar{I}} = LT^{-1} \text{ .... (25)}.$$

Capacity is defined as before by $\frac{\bar{Q}}{\bar{V}}$. Therefore

$$\bar{C} = \frac{M^{\frac{1}{2}}L^{\frac{1}{2}}}{M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}} = L^{-1}T^{2} \text{..............(26)}.$$

Strength of Electrical Field or *Electrical Force* is force on unit quantity, and is therefore measured by $\frac{\bar{F}}{\bar{Q}}$, which gives

$$\bar{F} = M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2} \text{.....................} (27).$$

For the strength of a Magnetic field the dimensions are still those of Art. 251 or $\bar{H} = M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$.

Number of Lines of Magnetic Force may (Art. 219) be defined by the formula $\bar{N} = \frac{m}{OA^2}\cos\epsilon\,.\,\sigma$. Hence

$$\bar{N} = m = M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \text{..................} (28).$$

Coefficient of Mutual or Self-Induction is defined (Arts. 238, 239) by the formula $Li^2 =$ energy of circuit; hence

$$\bar{L} = \frac{\bar{W}}{\bar{I}^2} = L \text{ ....} (29).$$

Specific Conductivity is defined as before by the relation

$$\bar{c} = \frac{\bar{I}}{\bar{F}L^2} = L^{-2}T \text{ ....} (30).$$

Specific Resistance is defined as the reciprocal of the last, and therefore

$$\bar{\rho} = L^2T^{-1} \text{........................} (31).$$

Specific Inductive Capacity may be defined by Art. 179, in agreement with the formula $\overline{RC} = \frac{1}{4\pi}\bar{\rho}\bar{K}$, whence

$$\bar{K} = \frac{T}{\bar{\rho}} = L^{-2}T^2 \text{ ....} (32).$$

253. We will now present in a tabular view the results of the preceding articles, adding the ratio between the dimensions of the various units where they have been measured in both the electrostatic and the electromagnetic systems, this ratio being the number of electrostatic in one electromagnetic unit.

(1) Mechanical.

Unit.	Symbol.	Dimensions.
Velocity	v	LT^{-1}
Acceleration ...	f	LT^{-2}
Force	F	MLT^{-2}
Energy	W	ML^2T^{-2}

254. (2) Electrostatic and electromagnetic units.

Units.	Symbol.	Dimensions in Electrostatic Measure.	Dimensions in Electromagnetic Measure.	Ratio of Electrostatic to Electromagnetic Measure.
Quantity	Q	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}$	LT^{-1}
Potential Electromotive Force	V E	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$L^{-1}T$
Electric Force	F	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}$	$L^{-1}T$
Capacity	C	L	$L^{-1}T^2$	L^2T^{-2}
Specific inductive capacity.	K	0	$L^{-2}T^2$	L^2T^{-2}
Specific conductivity	c	T^{-1}	$L^{-2}T$	L^2T^{-2}
Specific Resistance	ρ	T	L^2T^{-1}	$L^{-2}T^2$
Resistance of a Conductor..	R	$L^{-1}T$	LT^{-1}	$L^{-2}T^2$
Current-strength	I	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	LT^{-1}
Magnet pole Quantity of Magnetism	m	——	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	——
Magnetic potential	V	——	$M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$	——
Magnetic Force Strength of Field	H	——	$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$	——
Number of Lines of Force..	N	——	$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$	——
Coefficient of Mutual or Self-induction	L		L	

255. In the above table we see that the ratio of the measures of each magnitude in the two systems depends on LT^{-1}, a quantity of the dimensions of a velocity. This velocity has been determined by experiment, and must be an absolute quantity independent of any particular system of measurement. We will denote it by v, which may be assumed equal to 300 million metres per second or $3 \cdot 10^{10}$ cm. per second. From the preceding table we clearly have for the ratios between measures in the two systems

$$\frac{Q}{\bar{Q}} = \frac{I}{\bar{I}} = \frac{\bar{V}}{V} = \frac{\bar{F}}{F} = v,$$

$$\text{and } \frac{C}{\bar{C}} = \frac{K}{\bar{K}} = \frac{c}{\bar{c}} = \frac{\bar{\rho}}{\rho} = \frac{\bar{R}}{R} = v^2.$$

256. If the suffix 0 denote that the symbols stand for units instead of measures, we shall have for the ratio between the units themselves by Art. 248

$$\frac{Q_0}{\bar{Q}_0} = \frac{I_0}{\bar{I}_0} = \frac{\bar{V}_0}{V_0} = \frac{\bar{F}_0}{F_0} = \frac{1}{v},$$

$$\text{and } \frac{C_0}{\bar{C}_0} = \frac{K_0}{\bar{K}_0} = \frac{c_0}{\bar{c}_0} = \frac{\bar{\rho}_0}{\rho_0} = \frac{\bar{R}_0}{R_0} = \frac{1}{v^2}.$$

257. *Practical units* are not the absolute units given above, immediately derived from the C.G.S. system of measurement. It is found that by choosing these units, all our resistances and electromotive forces will be represented by very large numbers, and all our quantities and capacities by small fractions. The units of length, time, and mass actually taken are these:

For length, the quadrantal arc of the earth or 10^9 cm.

For time, the second remains the unit.

For mass, the 10^{-11} gm. is chosen.

Referring to the table of dimensions given above we see that electromotive force ($M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}$) becomes multiplied by $10^{-\frac{11}{2}} . 10^{\frac{27}{2}} = 10^8$. The *volt* is therefore defined as 10^8 absolute electromagnetic units.

Resistance, whose dimensions are LT^{-1} (the same as those of a velocity), becomes multiplied by 10^9. The *ohm* is defined as the resistance which is represented by the velocity of 10^9 cm. per second.

Current strength whose dimensions are $M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}$ is multiplied by 10^{-1}. The *ampere* or current strength due to E.M.F. of one *volt* acting through one *ohm* represents 10^{-1} absolute C.G.S. units.

Quantity, whose dimensions are $L^{\frac{1}{2}}M^{\frac{1}{2}}$, is multiplied by 10^{-1}. The Coulomb is defined as representing 10^{-1} units of quantity and is the quantity of electricity which flows per second in a current of one ampere.

Capacity, whose dimensions are $L^{-1}T^2$, is multiplied by 10^{-9}. The *farad*, which is the unit of capacity, will therefore represent 10^{-9} absolute C.G.S. units of the electromagnetic system.

The absolute unit of self-induction has for its dimension a simple length and the absolute unit would therefore be the self-induction in a circuit in which the coefficient is 1 cm.

The dimensions of resistance however are LT^{-1} and the absolute unit would be the resistance represented by a cm. per sec. The ohm however is represented by 10^9 cm. per sec., and it is found convenient to use as practical unit of self-induction that which is represented by 10^9 cm. This unit has been called by Prof. Ayrton the *secohm*, as it is clearly equivalent dimensionally to the ohm multiplied by the second, and it contains 10^9 absolute units of self-induction.

Thus we see that to convert into absolute electromagnetic units of the C.G.S. system,

the volt, the ohm, the ampere, the coulomb, the farad and the secohm we multiply respectively by

$$10^8,\ 10^9,\ 10^{-1},\ 10^{-1},\ 10^{-9},\ 10^{-9}.$$

We now see that the Watt which represents the rate of expenditure of energy represented by one ampere through one volt is measured by $10^{-1} \times 10^8 = 10^7$ ergs per sec.; as was assumed in Art. 80. On comparing the expression for the Horse Power in ergs per sec. (Art. 20) we see that 746 Watts are equal to one Horse Power.

In many cases very large multiples and submultiples of these units occur, and it is then convenient to use the prefix mega—or megal—to express a million times the unit and the prefix micro—to express a millionth part of the unit. Thus the megohm means 10^6 ohms and the microvolt means 10^{-6} of a volt.

Examples on Chapter IX.

1. Make the following conversions by means of the formulæ given in this chapter.

(1) Find the number of ergs in a foot-poundal.

Ans. 421383 nearly.

(2) Convert the acceleration of gravity (32·2) into the C.G.S. system. *Ans.* 981 nearly.

(3) The units of time, space, and mass, being changed from a second, foot, and pound to a minute, yard, and cubic foot of mercury (density 13); find the ratio of the new units of velocity, force and work, to the old units respectively.

Ans. $\frac{1}{20}, \frac{65}{96}, \frac{65}{32}$.

2. Show that in electromagnetic measure the dimensions of current-strength are given by $\frac{\text{magnet pole}}{\text{length}}$. Hence show that the electromagnetic attraction between two conductors carrying currents will be of the same dimension as the product of the current-strengths.

3. By comparing the attraction between two currents with that between two quantities of electricity condensed in points, show that the ratio between the electromagnetic and electrostatic units of quantity is represented by a velocity.

4. Show also that a current in electromagnetic measure is of the same dimensions as magnetic potential.

5. A sphere is raised to a given potential and discharged through a wire, the sphere contracting during the discharge so that its potential remains constant; show that the rate of contraction of the sphere will be equal to the reciprocal of the resistance of the wire in electrostatic measure.

6. Show that the heat given out in any circuit is expressed by the same formula whether the units be electrostatic or electromagnetic. Find also the formula if we use *practical* electromagnetic units.

7. A coil whose resistance is 2 ohms is immersed in a kilogramme of water and a current of 3 amperes passes through it for a minute. Find the elevation in temperature of the water (assuming the mechanical equivalent of heat to be 41560000 ergs per gm.-degree). *Ans.* $\frac{1}{4}^\circ$ C. nearly.

8. Find the radius of a sphere whose electrical capacity is one farad. *Ans.* 3×10^9 cm.

9. The electrostatic capacity per nautical mile of any gutta-percha cable is found to be $\dfrac{\cdot 18769}{\log \dfrac{D}{d}}$ farads, and the resistance of its insulating sheath $\dfrac{\log \dfrac{D}{d}}{13} \cdot 10^6$ ohms. Calculate the time of falling to half charge. (Given $\log_\epsilon 2 = \cdot 6931471$.)

Ans. 2 hrs. 46 min. nearly.

10. The resistance of gutta-percha is to that of Hooper's material as 1 to 16, and the specific inductive capacity as 4·2 to 3·1. Find from the last result the time of falling to half charge in a condenser of Hooper's material.

Ans. 32 hrs. 48 min.

11. Two plates whose areas are each one sq. cm. being placed at a distance of 2 mm. apart and connected with the terminals of a battery, are found to exert on each other a force equal to ·01 gm. Find in electrostatic and electromagnetic measure the electromotive force of the battery.

Ans. $3\cdot 1$ and $9\cdot 3 \times 10^{10}$ nearly.

12. Show that the electromotive force in the preceding question is nearly equal to that of 930 Daniell's cells.

13. A metre is defined to be a ten-millionth part of the quadrantal arc of the earth, express in absolute electrostatic and electromagnetic units the capacity of the earth.

Ans. $\dfrac{2 \times 10^9}{\pi} : \dfrac{2}{9\pi \times 10^{11}}$.

14. Find an expression for the force between two quantities of electricity q, q' expressed in electromagnetic measure placed D cm. apart, where $v = 3\cdot 10^{10}$ (Art. 255).

Ans. $\dfrac{1}{v^2}\dfrac{qq'}{D^2}$.

CHAPTER X.

PROBLEMS IN MAGNETISM.

258. THE following five propositions on the properties of bodies free to move about a fixed axis, which might have been placed in Chap. I., will be found of service in this chapter, when treating of the motion of suspended magnets in a uniform magnetic field.

259. The idea we have here to introduce is that of *angular motion*, which can be understood by fixing on a line in the body perpendicular to its axis and showing the angle traced out during the motion; the rate at which this angle is traced out being the angular velocity.

DEF. *The angular velocity of a rotating body is the angle traced out per second by a line fixed in the body perpendicular to the axis of rotation.*

The angular velocity like ordinary velocity is a property of the body at a particular instant, and if variable must be measured by the angle which would be traced out per second, supposing the angular velocity to remain constant.

Prop. I. If a body be rotating with angular velocity ω the velocity of a particle in the body distant r from the axis of rotation is $r\omega$.

For each particle traces out a circle, and if θ be the angle traced out in a small time τ, the angular velocity will be $\frac{\theta}{\tau}$; or $\theta = \omega\tau$. The length of the arc traced out by the particle is $r\theta$, and the velocity of the particle $\frac{r\theta}{\tau}$ or $r\omega$.

260. Prop. II. To find the energy of a body rotating with angular velocity ω.

The energy of the particle whose velocity we computed in the last article is $\frac{1}{2}mv^2$ when m is its mass, and this is equal to $\frac{1}{2}mr^2\omega^2$. Hence the energy of the whole body is

$$\frac{\omega^2}{2}\,.\,\Sigma mr^2.$$

Σmr^2 depends only on the density and shape of the body, and may be computed when the form of the body is known. It is defined to be the *moment of inertia*, and may be denoted by the symbol M. The energy of the rotating body we can then write $\frac{1}{2}M\omega^2$.

261. Prop. III. To find the angular velocity imparted to a body by a couple acting for a given time.

Let F be the force and l the arm of the couple. The work done by the force in twisting the body through a small angle θ is $F \times l\theta$.

If ω_1 and ω_2 be the angular velocities at the beginning and end of the movement the energy imparted is

$$\tfrac{1}{2}M(\omega_2^2 - \omega_1^2);$$

$$\therefore\ \tfrac{1}{2}M(\omega_2^2 - \omega_1^2) = F \times l\theta = G\frac{\omega_1 + \omega_2}{2}\tau,$$

supposing G the moment of the couple and τ the time occupied by the movement,

$$\therefore\ M(\omega_2 - \omega_1) = G\tau.$$

If the couple remain constant for any finite time t and ω be the whole angular velocity imparted

$$M\omega = Gt.$$

262. Prop. IV. To find the elongation of swing of a body acted on by a constant force in a given direction (e.g. a pendulum under gravity).

Let AB be the plumb-line in the position of equilibrium, and AC the limit of its swing, then CAB is the elongation required.

If B' be an intermediate position the work spent between B and B' will be $F\,.\,BN = Fl(1 - \cos\theta) = G(1 - \cos\theta)$, where $\theta = BAB'$ and $Fl = G$.

Hence if ω_0 be the angular velocity at B and ω at B',

$$\tfrac{1}{2}M(\omega_0^2 - \omega^2) = G(1 - \cos\theta).$$

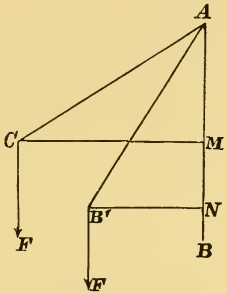

Fig. 81*.

If $CAB = \alpha$, then when $\theta = \alpha$, $\omega = 0$;

$$\therefore\ \tfrac{1}{2}M\omega_0^2 = G(1 - \cos\alpha) = 2G\sin^2\frac{\alpha}{2};$$

$$\therefore\ \omega_0 = 2\sin\frac{\alpha}{2}\sqrt{\frac{G}{M}},$$

which gives the relation required.

263. Prop. V. To find the time of oscillation about the position of rest of the body in the preceding article, the disturbing force being supposed small.

Let, as before, AB be the position of equilibrium, AC and AC' the extreme elongations.

For the angular velocity at any intermediate position AP, we shall have by the preceding Article

$$\tfrac{1}{2}M\omega^2 = G(\cos\theta - \cos\alpha),$$

where $\theta = PAB$ and $\alpha = CAB$.

If the disturbing force be very small the elongation will also be small, and we may put $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\cos\alpha = 1 - \frac{\alpha^2}{2}$;

$$\therefore\ M\omega^2 = G(\alpha^2 - \theta^2).$$

Set off the double elongation CBC' on a straight line so that $CC' = 2\alpha$, and $BP = \angle BAP$ and $BQ = \angle BAQ$. On CC'

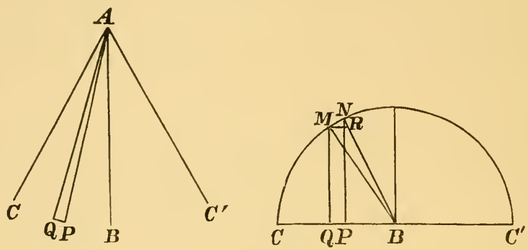

Fig. 82.

describe a semicircle and raise perpendiculars, PN, QM, and draw MR parallel to BC. Join BM, BN. Then

$$M\omega^2 = G(\alpha^2 - \theta^2)$$
$$= G(BN^2 - BP^2) = G \,.\, PN^2;$$
$$\therefore \omega = \sqrt{\frac{G}{M}} \,.\, PN.$$

Hence time of describing the arc PQ

$$= \frac{P\hat{A}Q}{\omega} = \frac{PQ}{\sqrt{\frac{G}{M}} \,.\, PN} = \sqrt{\frac{M}{G}} \cdot \frac{PQ}{PN}.$$

But by similar triangles $PQ : MN :: PN : BN$,

$$\therefore \frac{PQ}{PN} = \frac{MN}{BN} = \angle MBN;$$

$$\therefore \text{time of describing } PQ = \sqrt{\frac{M}{G}} \,.\, M\hat{B}N.$$

Adding all the successive intervals we shall have the time of describing $BC = \frac{\pi}{2}\sqrt{\frac{M}{G}}$.

Hence if T be the time of an oscillation from rest to rest

$$T = \pi\sqrt{\frac{M}{G}};$$
$$\therefore GT^2 = M\pi^2.$$

Cor. 1. We see from the result that the time of oscillation is independent of the arc of vibration, supposing this arc small. The vibrations are in consequence said to be isochronous.

When the arc of vibration is not very small the formula will not be rigorously true. We can easily find a superior limit to the error committed in using it.

Continuing the series for $\cos\theta$ and $\cos\alpha$ one term further we have

$$M\omega^2 = G\left(\alpha^2 - \theta^2) - \frac{2(\alpha^4 - \theta^4)}{\underline{|4}}\right)$$

$$= G(\alpha^2 - \theta^2)\left(1 - \frac{2(\alpha^2 + \theta^2)}{\underline{|4}}\right),$$

$$\therefore \omega = \sqrt{\frac{G(\alpha^2 - \theta^2)}{M}}\left(1 - \frac{\alpha^2 + \theta^2}{\underline{|4}}\right).$$

Hence the error in ω cannot exceed $\frac{\alpha^2 + \theta^2}{\underline{|4}}$ of the whole, and this is certainly less than $\frac{2\alpha^2}{\underline{|4}}$ or $\frac{\alpha^2}{12}$ of the whole, or the error in the time of an oscillation will certainly not exceed $\frac{\alpha^2}{12}$ of the whole time, α being half the angle of swing.

Cor. 2. If the pendulum consist of a mass m suspended from a weightless string of length l, under gravity $M = ml^2$ and $G = mlg$; hence $T = \pi\sqrt{\frac{l}{g}}$.

264. We now append some important applications of the preceding theory.

Prop. VI. To find the force acting on a magnet placed in a uniform magnetic field.

At a great distance from a magnetic system we may for a limited space consider the strength of the magnetic field to be uniform. The lines of force will then be a system of parallel lines uniformly distributed through the space. This will apply for instance in the case of the earth throughout

an ordinary room in which we perform our experiments. The direction of the lines of force are then shown by the dipping-needle, and the thickness of their distribution by the absolute value of the magnetic intensity at the place under consideration.

Let now the lines of force be parallel to the length of the paper, and let their direction be from the bottom to the top of the page.

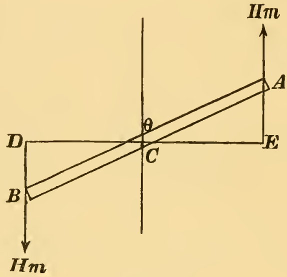

Fig. 83.

Let AB be a magnet whose north pole is at A and south pole at B, and let the strength of each pole be m. Let l be the length of the magnet, and θ its inclination to the lines of force.

Let H be the strength of the field, that is the force with which a unit pole would be urged along the lines of force. Then the pole A will be subject to a force Hm along the lines of force, and B will be subject to an equal and opposite force $-Hm$ along the lines of force.

These two forces constitute a couple (Art. 22), and the moment of the couple is $Hm \,.\, DE$.

But $$DE = l \sin\theta.$$

Hence the force on the magnet will be a couple whose moment is

$$Hml \sin\theta.$$

Cor. 1. If the magnet be placed perpendicular to the lines of force, the moment of the forces acting on it becomes Hml or the magnetic moment multiplied by the strength of the field (Art. 207).

Cor. 2. If the magnet be composed of several magnets with their axes all in the same direction, the moment of the couple on the compound magnet will clearly be $H \sin\theta . \Sigma ml$ when Σml is the sum of the moments of the elementary magnets, and may therefore be termed the moment of the compound magnet. This has already been assumed in (Art. 207) defining magnetic moment.

265. Prop. VII. To calculate the swing of a magnet placed in a magnetic field and making oscillations about its position of rest.

The calculation of the preceding articles applies, and if G be the magnetic moment, M the moment of inertia, ω the angular velocity in the position of rest, α the greatest elongation, and H the strength of the field, then (Art. 262)

$$M\omega^2 = 2HG(1 - \cos\alpha),$$

or $$\omega = 2 \sin\frac{\alpha}{2}\sqrt{\frac{GH}{M}}.$$

266. Prop. VIII. To find in absolute measure the magnetic moment of a given bar-magnet uniformly magnetized.

If G be the magnetic moment of the magnet, and H the horizontal component of the earth's magnetism, two observations are made, one of which gives the product HG, and the other the quotient $\frac{H}{G}$ from which G can be at once found.

First. To find the product HG. Let the magnet be suspended so as to swing freely in a horizontal plane, and let its time of oscillation be carefully observed. Then applying the formula of Art. 263, we have

$$HGT^2 = \pi^2 M \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where T is the observed time of oscillation in seconds, and M the moment of inertia of the magnet about its axis of suspension.

Secondly. By using the oscillation magnet of the last experiment and placing its axis East or West of a short suspended magnet we are enabled to obtain the ratio $\frac{G}{H}$.

Let a pole of the magnet of the last article of strength m be placed at A and let BC be another magnet disturbed by it from MM the plane of the meridian. Let the strength of the poles of BC be each m', its length $2c$, the distance of its middle point from A, d, and the distances AB, AC of the poles r_1 and r_2.

The force due to the pole acting on B will be $\frac{mm'}{r_1^2}$ along BA: the forces on C will be $\frac{mm'}{r_2^2}$ along AC.

The force $\frac{mm'}{r_1^2}$ in AB can be resolved into (Art. 17)

$$\frac{mm'}{r_1^2}\cdot\frac{d}{r_1} \text{ parallel to } AO, \quad \text{and } \frac{mm'}{r_1^2}\cdot\frac{c}{r_1} \text{ along } BO.$$

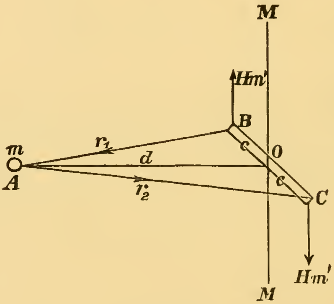

Fig. 84.

The force $\frac{mm'}{r_2^2}$ in AC can be resolved into

$$\frac{mm'}{r_2^2}\cdot\frac{d}{r_2} \text{ parallel to } AO, \quad \text{and } \frac{mm'}{r_2^2}\cdot\frac{c}{r_2} \text{ along } CO.$$

The moment of this system of forces about O, the suspension of the magnet BC

$$= c\cos\delta\left(\frac{dmm'}{r_1^3}+\frac{dmm'}{r_2^3}\right)$$

$$= dcmm'\cos\delta\left(\frac{1}{r_1^3}+\frac{1}{r_2^3}\right).$$

Now $$r_1^2 = c^2 + d^2 - 2cd \sin \delta$$
$$= d^2\left(1 - \frac{2c \sin \delta}{d} + \frac{c^2}{d^2}\right).$$

If we neglect all powers of $\frac{c}{d}$ above the first, assuming the magnet BC to be very short compared with AO, the distance of the deflecting pole

$$r_1^{-3} = d^{-3}\left(1 - \frac{2c \sin \delta}{d}\right)^{-\frac{3}{2}} = d^{-3}\left(1 + \frac{3c \sin \delta}{d}\right),$$

and similarly $$r_2^{-3} = d^3\left(1 - \frac{3c \sin \delta}{d}\right);$$

$$\therefore \frac{1}{r_1^3} + \frac{1}{r_2^3} = \frac{2}{d^3}.$$

Hence the moment $$= \frac{2cmm' \cos \delta}{d^2}.$$

If the other pole of the deflecting magnet be at distance e—remembering that it is of opposite sign—it will give the moment

$$-\frac{2cmm' \cos \delta}{e^2}.$$

$\therefore$ Total moment due to deflecting magnet

$$= 2cmm' \cos \delta \left(\frac{1}{d^2} - \frac{1}{e^2}\right).$$

This in the position of equilibrium will be equal to the moment of the earth's field on the magnet which is represented by Hm' at B and C parallel to MM;

$$\therefore 2cmm' \cos \delta \left(\frac{1}{d^2} - \frac{1}{e^2}\right) = 2Hm'c \sin \delta,$$

$$\therefore m\left(\frac{1}{d^2} - \frac{1}{e^2}\right) = H \tan \delta,$$

$$\frac{m(e-d)(e+d)}{d^2 e^2} = H \tan \delta,$$

or $$\frac{G(e+d)}{d^2 e^2} = H \tan \delta.$$

Since $$m(d-e)=G,$$

$$\therefore \frac{G}{H}=\frac{d^2e^2}{e+d}.\tan\delta\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Combining equations (1) and (2) we get the magnetic moment of the magnet and H the horizontal component of the earth's magnetism both in absolute measure. This method is due to the mathematician Gauss, and has the advantage of giving a result for H independent of the magnets employed; without which observations separated by a considerable interval are not comparable, since the magnetism of a magnet is generally undergoing slow alteration.

267. Prop. IX. To find the strength of field at any point due to a uniformly magnetized sphere.

We represent the distribution of magnetism (Art. 211) by two equal nearly coincident spheres, one of positive and the other of negative matter of density, ρ, having the line CC' joining their centres in the direction of magnetization, and such that $\rho\,.\,CC'=\mu$, the intensity of magnetization. The density of free magnetism will be given by $\mu\cos\theta$, where θ is the angle between the normal to the sphere and the direction of magnetization.

First, to find the strength of field at an external point P. We see that the forces on a unit pole at P will be the same as if all the positive sphere were collected in C, and all the negative sphere in C', the forces will therefore be $\frac{4}{3}\pi\rho\frac{a^3}{CP^2}$

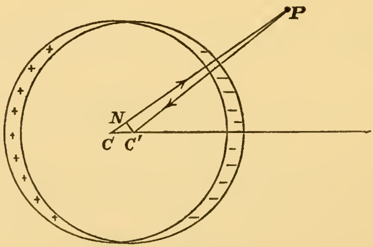

Fig. 85.

repulsive along CP, and $\frac{4}{3}\pi\rho\frac{a^3}{C'P^2}$ attractive and along $C'P$, a being the radius.

We will resolve these forces along and perpendicular to CP, calling the angles CP, $C'P$ make with CC' produced, θ, θ'. Then the component along CP outwards

$$=\tfrac{4}{3}\pi\rho a^3\left\{\frac{1}{CP^2}-\frac{\cos(\theta'-\theta)}{C'P^2}\right\}\ldots\ldots\ldots\ldots(1),$$

and the component perpendicular to CP increasing θ

$$=-\tfrac{4}{3}\pi\rho a^3\,.\,\frac{\sin(\theta'-\theta)}{CP^2}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Remembering that $\theta'-\theta$ is a small angle, the component along CP

$$=\tfrac{4}{3}\pi\rho a^3\left(\frac{1}{CP^2}-\frac{1}{C'P^2}\right)$$

$$=-\tfrac{4}{3}\pi\rho a^3\,.\,\frac{CP^2-C'P^2}{CP^2C'P^2}$$

$$=-\tfrac{8}{3}\pi\rho a^3\,.\,\frac{CP-C'P}{CP^3}\text{ very nearly.}$$

But if $C'N$ be perpendicular to CP,

$$CP-C'P=CN=CC'\cos\theta;$$

$$\therefore\text{ force along }CP=-\tfrac{8}{3}\pi\rho CC'\,.\,\frac{a^3}{CP^3}\cos\theta$$

$$=-\tfrac{8}{3}\pi\mu\frac{a^3}{CP^3}\cos\theta,$$

and force perpendicular to CP

$$=-\tfrac{4}{3}\pi\rho a^3\frac{\theta'-\theta}{CP^2}$$

$$=-\tfrac{4}{3}\pi\rho a^3\,.\,\frac{C'N}{CP^3}$$

$$=-\tfrac{4}{3}\pi\rho\,.\,CC'\frac{a^3}{CP^3}\sin\theta$$

$$=-\tfrac{4}{3}\pi\mu\frac{a^3}{CP^3}\,.\sin\theta.$$

If ϕ be the angle the line of force makes with CP

$$\tan\phi = \tfrac{1}{2}\tan\theta;$$

and the strength of field at P will be given by

$$\tfrac{4}{3}\pi\mu\frac{a^3}{CP^3}\sqrt{4\cos^2\theta+\sin^2\theta}.$$

Secondly, let P be internal. The force due to all of the sphere outside the given point vanishes (Art. 32), and force at P due to sphere of radius CP is $\frac{4}{3}\pi\rho CP$.

Hence there will be a repulsive force $\frac{4}{3}\pi\rho CP$ along CP, and an attractive force $\frac{4}{3}\rho C'P$ along PC'.

Hence by a triangle of forces the resultant will equal $\frac{4}{3}\pi\rho CC'$ in direction CC', or the force will everywhere be constant and equal to $\frac{4}{3}\pi\mu$.

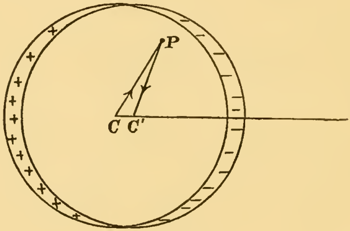

Fig. 86.

Cor. If there be a spherical cavity within the magnet, the principle of displacement of + and − fluids shows that the distributions over the internal and external surface will be according to the same law, but of opposite signs. Their total effect on any internal point will therefore be nil, or the force at every point within the cavity vanishes.

268. Prop. X. A sphere of perfect magnetic susceptibility placed in a uniform magnetic field will be uniformly magnetized.

By perfect susceptibility is meant a total absence of coercive force, so that any element at once takes up the magnetism due to the total force acting on it.

If H be the strength of field and k the coefficient of magnetization, let $Hk = \mu$. Suppose a sphere rigidly mag-

netized with intensity μ, placed with its lines of magnetization along the lines of force in the field. Make a small spherical cavity in this sphere. The force within it will be simply the force due to the external field. Fill the cavity with magnetic but unmagnetized matter, it will become magnetized to intensity Hk or μ, and will not be distinguishable from a portion of the original magnet. Make another spherical hollow, and go on repeating the whole process till as large a fraction as you please of the original magnet has been replaced by magnetic matter uniformly magnetized under the induction of the field.

269. The following perfectly general proposition in every field of force is placed here on account of its application to the magnetic problem which follows.

Prop. XI. In any field of force the strength increases from any point towards the centre of curvature of the line of force, through the point, and the rate of increase is measured by the product of the strength into the curvature of the line of force.

Let AB, CD be any two consecutive lines of force, we will assume in the plane of the paper, and let AC, BD be the lines in which the planes are cut by two equipotential surfaces. AC, BD will cut AB, CD at right angles, and if produced will intersect in O, the centre of curvature of the line of force.

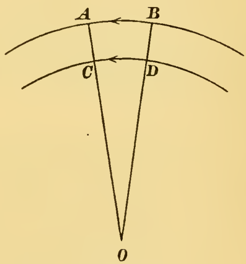

Fig. 87.

If F be the strength of field between A and B, and F' that between C and D, the difference of potential between AB and CD will

$$= F \times AB = F' \times CD.$$

But since AB is greater than CD, F' is greater than F.

Again
$$\frac{F'}{F} = \frac{AB}{CD} = \frac{AO}{CO};$$

$$\therefore \quad \frac{F'-F}{F} = \frac{AC}{CO}, \text{ or } \frac{F'-F}{AC} = \frac{F}{CO}.$$

Here, the left-hand side is the rate of change of F between A and C, and the right-hand side the product of F and $\frac{1}{CO}$, CO being the radius of curvature, and $\frac{1}{CO}$ therefore the measure of curvature.

270. Prop. XII. The resultant force on a small permanent magnet or small particle of magnetic matter placed along the lines of force in a magnetic field will be towards the strongest part of the field, and will be proportional to the rate of change of the strength in that direction.

Let NS be the magnet lying along a line of force, the positive direction of the line being from S to N.

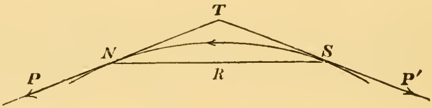

Fig. 88.

The forces acting on the magnet will be mP, mP' along tangents TN, TS drawn to the line of force at N, S. P, P' being the strength of field at N, S and m the strength of the poles N, S.

If θ be the angle subtended at the centre of curvature by the arc NS, the angles TNS, TSN will each equal $\frac{\theta}{2}$, and $\rho \, . \, \theta = NS$, where ρ is the radius of curvature.

Then resolving along NS we have

$$mP' \cos\frac{\theta}{2} - mP\cos\frac{\theta}{2}$$

$$= m(P'-P) = m \, . \, NS \, . \, \frac{P'-P}{NS}.$$

But $m \, . \, NS$ is the magnetic moment, and $\frac{P'-P}{NS}$ the rate of change of strength along the line of force.

The component towards the centre of curvature is

$$mP' \sin\frac{\theta}{2} + mP \sin\frac{\theta}{2}$$

$$= m.(P'+P).\frac{\theta}{2} = m.NS.\frac{P+P'}{2\rho}.$$

Here $\frac{P+P'}{2}$ is the average force between N and S, and therefore (Art. 269) $\frac{P+P'}{2\rho}$ is the rate of change of strength of field towards the centre of curvature.

Hence the components along and perpendicular to the line of force are each found by multiplying the magnetic moment by the rate of change of the strength in these respective directions.

It follows that the resultant force will be towards the strongest part of the field and will equal the magnetic moment multiplied by the rate of change of the strength in that direction.

Cor. 1. If the small magnet be placed in a reversed position, the positive direction of the lines of force being from N to S, the resultant force will have the same magnitude as before, but will be from the stronger to the weaker part of the field.

Cor. 2. If a small sphere of unmagnetized matter be placed in the field it will be magnetized along the lines of force, and the magnetic moment of the induced magnetism will be FkV where F is strength of field, k the coefficient of magnetization, and V the volume. Hence if F, F' be the strength of field at two points distant a apart, the average magnetic moment between these points is $kV.\frac{F+F'}{2}$, and, the resultant force on the sphere in the direction of a, which is assumed very small, is $kV.\frac{F+F'}{2}.\frac{F-F'}{a} = \frac{1}{2}KV\frac{F'^2-F^2}{a}$.

Cor. 3. The sphere of magnetic matter of the last article will be in equilibrium for movement in a given direction when

the rate of variation of F^2 vanishes, that is when F^2 is a maximum or a minimum. The equilibrium will be stable if F^2 is a maximum, that is at the strongest part of the field, and unstable if F^2 is a minimum, that is at the weakest part of the field. We have already seen that F^2 cannot be a maximum in free space, but only along a certain line on which the magnetized particle is constrained to move.

COR. 4. Since a piece of diamagnetic substance behaves like a steel magnet whose poles are reversed along the lines of force, if a small diamagnetic sphere be placed in a field of force, it will be urged by the force $\frac{1}{2}kV\dfrac{F'^2-F^2}{a}$, but from the strongest towards the weakest part of the field. This amounts to saying that the above formula is true always if for diamagnetic substances k be negative, a law we have already noticed.

COR. 5. The force between two diamagnetic masses or between different parts of the same diamagnetic are so weak that our means of experiment fail to detect them, and we may assume that every diamagnetic body is subject to forces which are the resultant of the forces which act on each element taken separately.

COR. 6. The behaviour of a diamagnetic in a magnetic field can generally be explained by the tendency of each element of its mass from places of strong to places of weak magnetic force. A bar of bismuth for instance would if undisturbed set along the line of force in a uniform field, but sets across them in the field of a strong electromagnet, where the strength falls off very rapidly as we recede across the field from the line joining the poles.

271. Prop. XIII. To find magnetic elements of a plane or helical voltaic circuit.

Let us consider first one turn of the circuit and let ρ be the density of the equivalent magnetic shell and d its thickness.

Then $i=\rho d$ by definition of the absolute electromagnetic units and therefore $\rho=\dfrac{i}{d}$.

Whether there be one turn or many $\frac{1}{d}$ will represent the number of turns in unit length of the helix, and if we call this n

$$\rho = ni.$$

Next, if the length of the helix be l, $ln = S$ where S is the total number of turns of wire and we get $\rho = \frac{Si}{l}$.

If A be the area of each turn, or the average area of the turns, if not all equal, the total quantity of magnetism on either end of the equivalent magnet is $A\rho = \frac{ASi}{l}$.

Since every unit of magnetism has 4π lines of force the total magnetic flux or number of lines of force embraced by the circuit $= 4\pi \frac{ASi}{l}$.

Thus we find for the strength of field anywhere within the helix, or the number of lines of force per unit area,

$$H = \frac{4\pi Si}{l} = 4\pi ni.$$

The magnetic moment of the helix is given by

$$A\rho \times l = ASi.$$

Cor. 1. A wire in the form of a helix traversed by a voltaic current may in all cases be substituted for a bar magnet, and as far as actions in the external magnetic field are concerned they will be identical.

The law of direction is seen at once if we remember that lines of force enter a magnet by its south pole, and leave

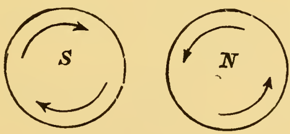

Fig. 89.

it by its north pole.

COR. 2. The result of the preceding Corollary may be used to determine a current in absolute electromagnetic measure.

For let a helix such as that considered be suspended by a bifilar arrangement perpendicular to the plane of the meridian, and let a current be transmitted through it by means of the wires of suspension; then if the Earth's force deflect it through an angle ϕ, the couple upon it will be

$$HA'i\cos\phi,$$

where H is the horizontal component of the Earth's magnetic force.

This must be balanced by the force of torsion in the suspending wires. If D be the coefficient of torsion this is measured by $D\sin\phi$;

$$\therefore\ HA'i\cos\phi = D\sin\phi;$$

$$\therefore\ i = \frac{D}{HA'}\tan\phi.$$

Let the current at the same time be passed through a tangent galvanometer in which there is a deflection ϕ'. Then, anticipating Art. 278,

$$i = \frac{H}{\Gamma}\tan\phi'.$$

Multiplying, we have

$$i^2 = \frac{D}{A'\Gamma}\tan\phi \,.\, \tan\phi';$$

$$\therefore\ i = \sqrt{\frac{D}{A'\Gamma}\tan\phi \,.\, \tan\phi'};$$

which gives the current-strength in absolute measure.

Eliminating i, we have

$$H = \sqrt{\frac{\Gamma D}{A'}\tan\phi \,.\, \cot\phi'},$$

another method of finding in absolute measure the Earth's Horizontal Force.

272. Prop. XIV. To investigate the properties of the electromagnetic field near a straight wire of infinite length carrying a current.

We may regard the infinite wire as part of a circuit, the rest of the circuit lying in a plane which we will assume perpendicular to the plane of the paper, as also the conductor under consideration which forms one edge of the circuit.

Let O be a section of the conductor, OA' the plane of the circuit, and P the given point.

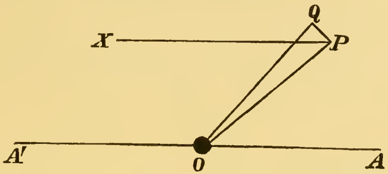

Fig. 90.

The circuit being of infinite extent above and below the paper and to the left of O, will now subtend a solid angle at P, which has for its boundary a plane passing through the conductor (whose projection is PO) and a plane passing through the opposite part of the circuit. This part is parallel to the given conductor and at an infinite distance, and the plane passing through it will be parallel to the plane of the circuit and will have for its projection PX, which is parallel to OA'.

The solid angle bounded by the two planes PO, PX will be a lune of the unit sphere, and its area will clearly be

$$\frac{\text{circular measure of } O\hat{P}X}{2\pi} \times \text{area of sphere}$$

$$= \frac{\theta}{2\pi} \times 4\pi = 2\theta,$$

where $$\theta = \angle POA.$$

Hence the potential at P is $2\theta i$.

This expression shows that for all points on the plane OP the potential is the same. Hence the equipotential surfaces are a system of planes intersecting in the conductor,

and the lines of force are consequently a system of circles in planes parallel to the paper having O for their centre.

We must also remember that each of these surfaces has not a definite potential $2\theta i$, but its general potential will be

$$(\pm 4n\pi + 2\theta)\, i.$$

To find the strength of the field or the magnetic force at P, we must find the rate of change of potential along a line of force. If PQ be an arc of a circle whose centre is O, it will be an arc of the line of force through P. If H be strength of field,

$$H = \frac{\text{Potential at } Q - \text{Potential at } P}{PQ}$$

$$= \frac{2P\hat{O}Qi}{PQ} = \frac{2i}{OP}; \text{ since } OP \,.\, P\hat{O}Q = PQ.$$

This gives us the strength of the field at every point round the conductor; the direction of the force being always perpendicular to a plane containing the conductor and the magnet pole.

Cor. 1. To find the attraction between the infinite conductor of this problem and another of finite length placed parallel to it and carrying a current.

Let O' be the trace on the paper of the second conductor of length l which will also be perpendicular to the paper, and let it carry a current i'.

If the conductor be moved parallel to itself to O'', the increase in the lines of force enclosed will be $H \times l \times O'O''$, when H is the strength of the field. Here $H = \dfrac{2i}{OO'}$;

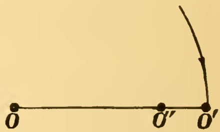

Fig. 91.

$\therefore$ the work done in the movement from O'' to O' (Art. 233)

$$= \frac{2il \,.\, O'O''}{OO'} i' = \frac{2lii'}{OO'} \,.\, O'O''.$$

But this equals $F.O'O''$ where F is the attraction between the two conductors; and therefore $F = \frac{2lii'}{OO'}$.

Cor. 2. The induced current in the second conductor owing to moving it parallel to itself, assuming that in the primary unaltered by the movement, is (Art. 244)

$$= -\frac{1}{R}\Sigma\frac{2liO'O''}{OO'} = +\frac{2li}{R}\Sigma\log\left(1-\frac{O'O''}{OO'}\right)$$

$$= \frac{2li}{R}\Sigma(\log OO'' - \log OO'),$$

R being the resistance in the circuit.

Hence during any finite movement, as from O_1 to O_2,

$$[i] = -\frac{2li}{R}\log\frac{OO_2}{OO_1}.$$

273. Prop. XV. To investigate the magnetic field along the axis of a circular voltaic circuit.

Let O be a point on the axis and CA the plane of the circuit.

To find the potential at O we have only to compute the solid angle subtended by the circuit at O. This will be the solid vertical angle of a right cone whose base is the circuit.

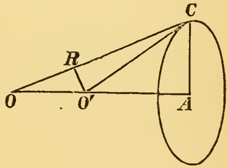

Fig. 92.

It is easy to see that the area of the unit sphere cut off by this cone is $2\pi(1-\cos\theta)$ where $\theta = C\hat{O}A$.

The potential at O is consequently $2\pi i(1-\cos\theta)$.

If the point be off the axis the cone becomes oblique, and there is no means of estimating its solid angle exactly.

To find the strength of the field at O we must compute the rate of change of potential in the direction OO'. This will be

$$\frac{2\pi i(\cos\theta - \cos\theta')}{OO'},$$

where $CO'A = \theta'$.

To exhibit this geometrically, draw OC' parallel to $O'C$ and describe a circular arc CC'. Draw CA, $C'A'$ perpendicular to OA, CE perpendicular to $C'A'$, and $O'D$ perpendicular to OC'.

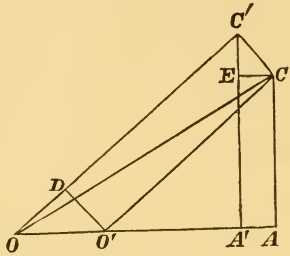

Fig. 93.

$$\text{Then } \cos\theta - \cos\theta' = \frac{OA}{OC} - \frac{OA'}{OC'} = \frac{OA - OA'}{OC} = \frac{AA'}{OC}$$

$$= \frac{CE}{OC} = \frac{CE}{CC'} \cdot \frac{O'D}{OO'} \cdot \frac{OO'}{OC},$$

remembering that CC' is approximately parallel to $O'D$.

But by similar triangles $\dfrac{CE}{CC'} = \dfrac{AC}{O'C}$ and $\dfrac{O'D}{OO'} = \dfrac{C'A'}{OC'}$.

Hence $\dfrac{2\pi i(\cos\theta - \cos\theta')}{OO'} = 2\pi i \dfrac{AC}{O'C} \cdot \dfrac{A'C'}{OC'} \cdot \dfrac{1}{OC}$, which when O, O' are very near becomes $\dfrac{2\pi i AC^2}{OC^3}$ the strength of the magnetic field at O.

Cor. 1. The potential at the centre is $2\pi i$, and the strength of field there is $\dfrac{2\pi i}{AC}$.

Cor. 2. If the wire be wound n times round the circle we consider separately the strength of field due to each coil and add them. The result will be that the potential and strength of field are each multiplied by n.

274. Prop. XVI. To investigate the Magnetic Flux in an electromagnet, consisting of a cylindrical core of iron surrounded by a wire helix carrying a current.

We have seen (Art. 271) that the strength of field within a helix carrying a current is given by

$$H=\frac{4\pi Si}{l},$$

and if we substitute iron whose permeability is μ for air we have (Art. 274)

$$B=\mu H=\mu\,.\,\frac{4\pi Si}{l}.$$

We must remember however that to be strictly comparable we must substitute iron for air both inside and outside the helix. This equation will however be true for an electromagnet with a massive iron armature connecting its poles: In this case the total Magnetic Flux is given by

$$AB=4\pi\mu\,.\,\frac{ASi}{l}$$

$$=4\pi\,.\,\frac{Si}{\frac{1}{\mu}\cdot\frac{l}{A}}.$$

or if i is to be measured in amperes we must divide by 10 and get for the flux

$$AB=\frac{4\pi}{10}\cdot\frac{Si}{\frac{1}{\mu}\frac{l}{A}}.$$

In this expression the numerator Si depends entirely on the number of magnetizing coils and the current strength and may be called the number of ampere turns. This, multiplied by $\frac{4\pi}{10}$, represents the power the current in the circuit possesses of forcing lines of magnetic force through its area, and is hence called the magnetomotive force from the analogy of the electromotive force in current electricity.

The expression $\frac{1}{\mu}\frac{l}{A}$ depends on the geometry and material

of the magnetic circuit and is of the same form as the resistance to an electric current of a conductor of specific conductivity μ, length l and cross-section A. It is hence called the magnetic resistance or reluctance of the magnetic circuit.

The equation written above is then identical in form with Ohm's formula for electric conduction.

In both cases if all the circuit, except the portion defined above by the symbols μ, l, A, were of very large sectional area and of high conductivity or permeability, the resistance or reluctance of the whole circuit is expressed by the term

$$\frac{1}{\mu}\frac{l}{A}.$$

Experiment shows that a very narrow space filled with air or other non-magnetic material makes a great difference in the reluctance of the circuit and a proportional diminution of the flux. Thus it becomes necessary to compute the reluctance of a magnetic circuit by estimating the length and section in each separate material traversed by the lines of force.

Thus we have a general expression for the magnetic flux

$$= \frac{4\pi}{10}\,\frac{Si}{\dfrac{1}{\mu_1}\dfrac{l_1}{A_1} + \dfrac{1}{\mu_2}\dfrac{l_2}{A_2} + \dfrac{1}{\mu_3}\dfrac{l_3}{A_3} + \text{\&c.}},$$

where the successive terms in the denominator represent the reluctance computed for each portion of a magnetic circuit. If one term belongs to air or to any non-magnetic substance, while the rest refer to varieties of iron μ becomes in that term unity and in all the others a very large number, and the one term $\frac{l}{A}$ though l is very moderate and A a large area may largely exceed the sum of all the other terms put together.

To produce therefore in practice a large magnetic flux we must make a magnetic circuit of pieces of iron of large section accurately fitted without air spaces at joints, while all air spaces which are absolutely necessary must be made of as large section and as thin as possible.

These principles are illustrated in the study of the form taken by the field magnets in modern dynamos.

Cor. We can easily compute the portative power of a magnet in which the magnetic flux is known.

Let B be the magnetic flux per unit area and A the area of the common surface of the armature and magnet (B must of course be computed when the armature is in position).

The density of the magnetism on the opposed surfaces of magnet and armature is $\pm\frac{B}{4\pi}$ and the quantity of magnetism on either is $\pm\frac{AB}{4\pi}$. Again the strength of field close to a distribution of attracting matter of density ρ is $2\pi\rho$ (Art. 36), and this becomes $\frac{B}{2}$ for the attraction of the magnet for each unit of magnetism close to its surface.

Hence the attraction exerted on the armature itself is $\frac{B}{2}$ for each unit of magnetism on the armature. Thus the total pull on the armature is

$$\frac{B}{2}\times\frac{AB}{4\pi}=\frac{AB^2}{8\pi}.$$

275. Prop. XVII. To find the coefficient of self-induction for a coil and of mutual induction for two coaxial coils.

We have seen that the number of lines of force embraced by a circuit in the form of a coil carrying a current i is

$$\frac{4\pi ASi}{l}.$$

If there be but a single turn of wire, the magnetic flux becomes

$$\frac{4\pi Ai}{l}.$$

Making $$i=1,$$

we get (Art. 239) $$L=\frac{4\pi A}{l}.$$

The intrinsic energy of the circuit $= \frac{1}{2} L i^2$.

If there be S turns each carrying a current i the intrinsic energy

$$= \tfrac{1}{2} L \,.\, (Si)^2$$

$$= \tfrac{1}{2} \,.\, \frac{4\pi A S^2}{l} \,.\, i^2.$$

Hence putting $i = 1$, the coefficient of self-induction

$$L = \frac{4\pi A S^2}{l}.$$

This of course gives the self-induction in absolute measure and to reduce it to secohms we must divide by 10^9 (Art. 257).

Let us next consider the two coaxial cylinders A and B. Let A, S, l represent the constants for the coil A and B, S', l' the corresponding constants for B. Let also the currents be i and i'.

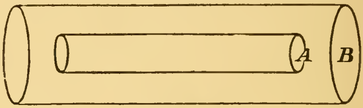

Fig. 94.

The total flux through B

$$= \frac{4\pi B S' i'}{l'}.$$

The number of lines enclosed by A

$$= \frac{4\pi A S' i'}{l'}.$$

If $S' = 1$ and $i' = 1$ the number of B's lines enclosed by A

$$= \frac{4\pi A}{l'}.$$

But, as before, if the currents were Si and $S'i'$ respectively the mutual energy

$$= \frac{4\pi A}{l'} \, Si \, S'i'$$

$$= \frac{4\pi A S S'}{l'} \, ii'$$

$$\therefore \; M = \frac{4\pi A S S'}{l'}.$$

For the case in which A has an iron core, unless A fits closely into B many of A's lines of force will return again within B, and these should be subtracted in computing the number enclosed by B. But in the case in which A fits tightly in B, the total flux through A is given by

$$4\pi Si \,.\, \frac{\mu A}{l},$$

where $\dfrac{l}{\mu A}$ is the reluctance of the magnetic circuit.

Similar reasoning to that adopted above shows that for A the self-induction

$$L = 4\pi S^2 \,.\, \frac{\mu A}{l}$$

and for the mutual induction

$$M = 4\pi SS' \frac{\mu A}{l}.$$

COR. The arrangement last referred to gives us the theory of the induction coil in which A is the primary and B the secondary circuit.

The working of the coil consists in making and breaking contact in A many times ($= n$ suppose) a second.

Each $\dfrac{1}{n}$ second the magnetic flux represented by

$$4\pi Si \,.\, \frac{\mu A}{l}$$

is driven through A and driven out again by breaking contact. Thus the time occupied in each process will be on an average $\dfrac{1}{2n}$ second.

Each $\dfrac{1}{2n}$ second the number of lines of force included in each turn of B will be altered by $4\pi Si \,.\, \dfrac{\mu A}{l}$ and the *average rate* at which lines of force embraced by B are changing is

$$2n \times 4\pi Si \frac{\mu A}{l} \times S' = 8\pi nSS'i \,.\, \frac{\mu A}{l}.$$

This will therefore represent the average E.M.F. of each of the induced currents called up in B. The direction of the current will be inverse when the flux is increasing and direct when it is declining.

This solution is of course only approximate, especially because we have taken no account of the self-induction which in a rapidly rising and falling current is of importance.

If this theory were complete it would appear that the E.M.F. of the induced currents depends only on the product S and S', the number of turns in the primary and secondary. But there are many reasons why S is always kept small and S' made very large. In the first place more battery power will be required to send the same current through many coils than through few: in the second place self-induction in the primary must be kept small since by diminishing the rate at which changes are taking place in the magnetic field it lowers the E.M.F. of the secondary: and thirdly it is only to a very moderate degree of magnetization in the core that the magnetization is proportional to the magnetizing force, the increase in magnetization being very small compared to that in the magnetizing force as magnetic saturation point is approached.

276. Prop. XVIII. To find the constant of a Galvanometer: that is to say, the strength of field close to the magnet poles due to unit current in the galvanometer coils.

If the needle be exceedingly short we may assume that the strength of field at either pole is the same as at the point of suspension halfway between them. If the galvanometer coil consist of a few (S) turns of wire nearly in the same plane the strength of field due to current i at the centre (Art. 273)

$$= \frac{2\pi Si}{a},$$

where n is the number of turns and a the average radius. Thus denoting the galvanometer constant by Γ we shall have making $i = 1$

$$\Gamma = \frac{2\pi S}{a}.$$

If the magnet is placed along the axis of the coil at a distance x from its centre

$$\Gamma = \frac{2\pi S a^2}{(a^2+x^2)^{\frac{3}{2}}} \text{ (Art. 273).}$$

The field will in this case of course not be uniform, the movement of the magnet bringing its poles into parts of the field of varying strength. Helmholtz has introduced galvanometers with two or three parallel circles having the needle hung symmetrically between them. In this case the galvanometer constant is found by adding the strength of field due to each separate circle as given by the last equation.

In instruments of the Mirror galvanometer type the strength of field within the coil may be taken as the average given in Art. 271 and we may write

$$\Gamma = \frac{4\pi S}{l},$$

S being the total number of turns and l the length of the cord.

Cor. 1. If Γ be the galvanometer constant, the strength of the field near the magnet will be Γi when a current i circulates in it, and if we substitute Γi for H the previous propositions about magnet motion are applicable.

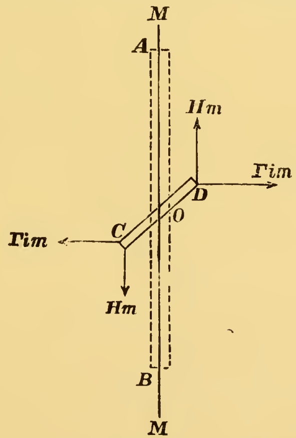

Fig. 95.

Cor. 2. If the plane of the galvanometer coil AB be the plane of the meridian, the strength of the current will be proportional to the tangent of the deflection of the magnet.

Let CD be the magnet, and let the strength of each pole be m. Also let H be the earth's horizontal force.

The forces acting on the poles will be $\pm Hm$ parallel to the magnetic meridian due to the Earth's magnetism, and Γim perpendicular to the meridian due to the voltaic circuit.

Hence taking moments about O, we have, if δ be the deflection AOD,

$$\Gamma im \,.\, CD \cos\delta = Hm \,.\, CD \sin\delta\,;$$

$$\therefore\; i = \frac{H}{\Gamma}\tan\delta.$$

This form of galvanometer is called a Tangent Galvanometer.

Cor. 3. If the coil be moveable about a vertical axis, and be turned round so that the magnet is in the plane of the coil, the current-strength is proportional to the sine of the deflection. For using the same notation as before and taking moments about O,

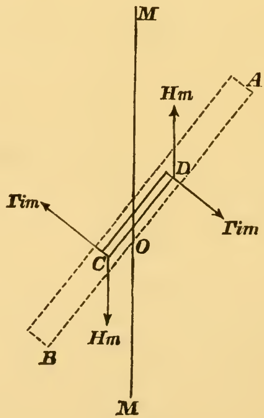

Fig. 96.

$$\Gamma im \, . \, CD = Hm \, . \, CD \sin \delta ;$$
$$\therefore \; i = \frac{H}{\Gamma} \sin \delta .$$

This form of the instrument is called a Sine Galvanometer.

The advantage of this over the Tangent Galvanometer is that the reading is always made with the needle in the same position relatively to the coil. The needle may therefore be made as long as we please and the coil may be a flat rectangular coil close to the needle. The constant Γ denotes the strength of the coil's field at the magnet pole, and may be determined by comparison with a Tangent Galvanometer.

277. It may be worth while to consider some rules as to the construction of galvanometers best adapted for different circuits. In the first place the galvanometer should be sensitive, that is to say, there should be as large a deflecting couple as possible when traversed by a given current. This couple depends on the strength of field Γi and the strength of the fixed magnetic field due to the earth or permanent magnets. Hence we must make Γ as large as possible.

In all cases of Art. 276 we notice that for a galvanometer coil of given dimensions Γ is simply proportional to the number of windings (S); which again is proportional to the length of wire used in the coil.

Suppose now we have a certain weight of copper for use, but have the gauge of wire at our disposal in making the galvanometer. We observe that if g be the resistance, l the length, s the cross-section and ρ the specific resistance, $g = \rho \frac{l}{s}$.

Also $ls = v$ the volume, which is constant for a constant weight of metal;

$$\therefore \; \frac{1}{s} = \frac{l}{v}, \text{ and } g = \frac{\rho l^2}{v}, \text{ or } l \propto \sqrt{g}.$$

This shows that for a galvanometer of given size and form placed in a given field, with given magnet system, and wound with a certain weight of copper, the galvanometer constant is proportional to the square root of the resistance.

But in a circuit the current strength is usually given by an equation of the form

$$i = \frac{E}{R+g},$$

where R is the resistance in the circuit outside the galvanometer.

Hence Γi or the deflecting couple will be the greatest when $i\sqrt{g}$ or $\dfrac{E\sqrt{g}}{R+g}$ is a maximum.

Now a similar investigation to that for greatest current in Art. 171 shows that this condition is satisfied when $R=g$.

Thus, with the same limitation, that is leaving only the gauge of wire disposable, the best galvanometer of given pattern is that whose resistance equals the rest of the resistance in the circuit.

A good example is afforded by the galvanometer for use in Wheatstone's Bridge. It is necessary that when the adjustment $qr=ps$ is not exactly made there shall be a maximum current in the galvanometer. The equation for the current in this branch is (Art. 183)

$$I_5 = \frac{(qr-ps)\,E}{R\,(p+r)\,(q+s)+pr\,(q+s)+qs\,(p+r)} \\ +g\,\{R\,(p+q+r+s)+(p+q)\,(r+s)\}.$$

To make the galvanometer as sensitive as possible $I_5\sqrt{g}$ must be a maximum: and this will be the case, as before, when

$$g\,\{R\,(p+q+r+s)+(p+q)\,(r+s)\} \\ = R\,(p+r)\,(q+s)+pr\,(q+s)+qs\,(p+r).$$

Now we wish the galvanometer to be sensitive when $qr=ps$ and we will find the value of g when that condition is satisfied. Writing $q\dfrac{r}{s}$ for p we find the equation reduces to

$$g\,(r+s)\,\{R\,(q+s)+q\,(r+s) \\ = r\,(q+s)\,\{R\,(q+s)+q\,(r+s)$$

whence $$g = \frac{r\,(q+s)}{r+s},$$

the value required for the best galvanometer assuming the gauge of wire as before only to be variable.

The above expression by a little algebraical transformation is equivalent to

$$\frac{(p+r)(q+s)}{p+q+r+s}.$$

278. Prop. XIX. To find the throw of a galvanometer needle owing to the passage of an instantaneous electric discharge through it.

If the strength of current at any instant be i, and last for a short interval τ, we have by Art. 261

$$M(\omega'-\omega)=G\Gamma i\tau,$$

M being the moment of inertia of the magnet, Γi the strength of the field, and G the moment of the magnet.

But $i\tau=q$ the quantity transmitted;

$$\therefore M(\omega'-\omega)=G\Gamma q.$$

If the magnet needle be fixed in a heavy ball of metal its inertia becomes so great that it does not move perceptibly from its position of rest while the transient current lasts. Hence the field will during the whole discharge be perpendicular to the magnet. In this case we can add both sides of the last equation during the whole discharge, and we have, if ω_0 be the impulsive angular velocity and Q the quantity transmitted,

$$M\omega_0=G\Gamma Q.$$

But by Art. 265 if an angular velocity ω_0 be imparted to the magnet, and if α be the throw of the needle,

$$\omega_0=2\sin\frac{\alpha}{2}\sqrt{\frac{HG}{M}};$$

$$\therefore G\Gamma Q=2\sin\frac{\alpha}{2}\sqrt{HGM},$$

$$\text{or } Q=2\frac{H}{\Gamma}\sqrt{\frac{M}{HG}}\cdot\sin\frac{\alpha}{2}.$$

If T be the time of a single vibration of the needle under the Earth's magnetism,

$$T=\pi\sqrt{\frac{M}{HG}};\ \text{(Art. 266)}$$

$$\therefore Q=\frac{2H}{\Gamma}\cdot\frac{T}{\pi}\cdot\sin\frac{\alpha}{2}.$$

If the constants H, Γ, T, are known, this equation gives us a means of measuring any electrical accumulation.

This galvanometer, called a ballistic galvanometer, is used for the measurement of all movements of electricity of short period. Thus it can be used to measure the flow of electricity from a battery in charging a condenser, and in practice the galvanometer is calibrated by such an observation since Γ and T are seldom accurately known. If, however, a condenser of standard capacity be charged from a battery whose E.M.F. is known, the charge is known and the throw of the galvanometer for this charge being known the charge corresponding to any observed throw can be found.

The galvanometer also enables us to measure a number of lines of magnetic force called up in a field by a current, for (Art. 244) the total flow of electricity through a circuit owing to any change in the magnetic field is given by

$$-\frac{\text{Number of lines of Force added}}{\text{Resistance of circuit}};$$

hence, observing the quantity of electricity transmitted through the galvanometer and knowing the resistance of the galvanometer and rest of the circuit, the number of lines of force added to or subtracted from the field of any coil connected with the galvanometer can be found.

279. An application of the foregoing principle is found in the practical method of finding the permeability of a given specimen of iron. First take a long helix of wire coiled round a cylinder of some non-magnetic substance, such that the magnetic flux can be easily calculated by the formula (Art. 274) $H=\frac{4\pi}{10}\cdot\frac{ASi}{l}$, when the current is measured in

amperes. Wind round the outside of the helix a measured number of turns (S') of wire and connect its terminals with those of a Ballistic galvanometer. On breaking contact with the battery we observe the throw which corresponds to the known number of lines of Force $\frac{4\pi}{10} \cdot \frac{ASS'i}{l}$.

Take now a sample of iron turned by a lathe into a solid ring such that the cross-section (A) and average circumference (l) can be formed by measurement. Coil a known number of turns (S) of wire round it and connect with a battery through a galvanometer by which the current (i) can be read in amperes. The magnetic flux is now represented by

$$\mu \cdot \frac{4\pi}{10} \cdot \frac{SiA}{l}$$

where μ, the permeability, is the only unknown. Wind round the iron ring a few turns (S') of wire as before and connect the terminals with the galvanometer. On breaking contact in the battery circuit we observe the throw which (remembering that the resistance of the coil is negligible compared with that of the galvanometer) is proportional to the number of lines enclosed. Thus we know the number of lines represented by $\mu \, \frac{4\pi}{10} \cdot \frac{ASS'i}{q}$, the symbols not necessarily having the same values as in the preceding expression. By a simple proportion the value of μ becomes known.

Another method depending on portative power deserves notice: It is an invention of Prof. Sylvanus Thompson, called the Permeameter, and is a workshop method for comparison of different samples of iron. The iron is turned in a bar of constant diameter which fits accurately into a helix of wire through which a measured current is sent. The helix and core slide into a cylindrical hole in a large block of iron, the bottom of the hole being made true, so that, when the helix is slipped in, the end of the core makes contact all over.

The measured current is now passed and the power required to draw out the core against the attraction of the

iron at its base is measured by a spring balance. This pull reduced to absolute measure (Art. 274) equals $\frac{B^2A}{8\pi}$ and A being known B becomes known. In comparing specimens of iron the cores are all turned up to the same dimensions, and the magnetizing current is kept constant. The values of B are then directly proportional to the permeabilities, which are seen to be simply as the square roots of the forces required to withdraw the core.

280. Prop. XX. To explain the action of the Dead-beat Galvanometer.

In a galvanometer with the needle swinging inside the coils, the movement of the poles produces an induced current which 'damps' the swing, or, in other words, produces a field whose action on the needle opposes its movements.

Since the two poles move in exactly opposite directions, their separate effects will be simply added. The strength of field produced by the induced current is found to be increased by increasing the number of turns in the galvanometer coil, and if this number be made great enough it may entirely check the free vibration of the magnet about its position of rest after the electromotive force producing the first elongation has sunk to zero (see Art. 243). The consequence of this will be, that as the needle returns from its first elongation, the motion is so much damped that it merely returns slowly to its position of rest, never passing it, so that the motion ceases to be one of oscillation.

This form of galvanometer is extremely useful in marine telegraphy, as it would be highly inconvenient to wait for the needle's return to rest between two consecutive signals.

These galvanometers are very expensive, owing to the enormous number of winds required in the wire coil. Their resistance is often as much as 30,000 or 40,000 ohms.

281 a. Prop. XXI. To explain the action of a Dynamo.

In a dynamo a coil of wire forming part of an armature is made to pass from a position in which it embraces a large number of lines of force generated by field magnets, to a

position in which the lines of force pass through it in the opposite direction and back again to a position in which it embraces the greatest number of lines. This alternation is made to occur many times a second, generally by the rotation either of the armature or of the field magnets.

During each quarter of a complete period, there is a change of magnetic flux through each loop of the coil equal to the total number of lines of force embraced by it, and there is an induced current in the coil whose direction reverses each half-period.

Suppose the flux of lines of force when at maximum to be represented by N, and let the number of turns in a coil be S, and suppose there are n complete periodic changes in a second.

Every $\frac{1}{4n}$ of a second there is a change in the lines of force represented by SN, or at the *average rate* of $4nSN$ lines of force per sec.

This then represents the *average* E.M.F. (Art. 246) in the circuit. This is of course in absolute measure, but can be reduced to volts by dividing by 10^8 (Art. 257). Hence the average E.M.F. in volts is given by

$$\text{(average) } E = 4 \,.\, 10^{-8} \,.\, nSN \ldots\ldots\ldots\ldots\ldots (1).$$

If the coil be rotating uniformly in a uniform magnetic field and N be the maximum flux through each of the S loops of the coil, the flux in any other position is given by $SN \cos\theta$, where θ is the angle turned through, and remembering that $\frac{1}{n}$ sec. is in this case the time of a complete rotation, $\theta = 2\pi nt$, where t is the time in which the coil turns through the angle θ.

Hence the magnetic flux at time $t = SN\cos 2\pi nt$,

and $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; t+\tau = SN\cos 2\pi n(t+\pi)$.

$\therefore$ Loss of lines of force in time τ

$$= SN\{\cos 2\pi nt - \cos 2\pi n(t+\pi)\}$$

$$= 2SN \sin 2\pi n\left(t+\frac{\tau}{2}\right)\sin \pi n\tau.$$

$\therefore$ Rate of change of magnetic flux

$$= 2SN \cdot \sin 2\pi n \left(t + \frac{\tau}{2}\right) \cdot \frac{\sin \pi n \tau}{\tau}$$
$$= 2\pi n SN \sin 2\pi n t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

remembering that τ is a very small interval.

Thus we have for E.M.F. in volts

$$E \text{ (at time } t) = 2 \cdot 10^{-8} \cdot \pi n SN \sin 2\pi n t \quad \ldots\ldots \quad (3),$$

or if θ be the position angle of a coil

$$E \text{ (in position } \theta) = 2 \cdot 10^{-8} \cdot \pi n SN \sin \theta \quad \ldots\ldots \quad (4).$$

This shows that the E.M.F. is smallest when the flux is a maximum, i.e. when the coil is at right angles to the magnetic field, and is greatest when the flux vanishes, that is in a position at right angles to the former.

If we notice that the sine of $2\pi nt$ goes through all its values while $2\pi nt$ changes from 0 to $\frac{\pi}{2}$, and take the average of all its values between these limits we shall find it to be $\frac{2}{\pi}$*. Putting this average value for the sine we observe that (2) becomes identical with (1).

* To prove this proposition we assume the right angle divided into n, a large number of very small angles each equal to α, so that $n\alpha = \frac{\pi}{2}$.

The average value of the sine

$$= \frac{1}{n}(\sin \alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha)$$
$$= \frac{1}{n} \cdot \frac{\sin \frac{n+1}{2}\alpha \cdot \sin n\frac{\alpha}{2}}{\sin \frac{\alpha}{2}}$$
$$= \frac{\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \sin \frac{\pi}{4}}{n \sin \frac{\alpha}{2}}$$
$$= \frac{\sin^2 \frac{\pi}{4}}{\frac{n\alpha}{2}} = \frac{\frac{1}{2}}{\frac{\pi}{4}} = \frac{2}{\pi},$$

remembering that α may be as small as we please by making n sufficiently great.

This theory is incomplete, since we have wholly omitted all effects of self-induction, which is a very important factor, where we have a rapid reversal of E.M.F. and of current as in the coil we have been discussing. In a direct current dynamo in which coils are so disposed in the armature as to give a nearly uniform E.M.F. and therefore current in the external circuit this theory is sufficient for all purposes, but in alternating current machines allowance for self-induction is imperative.

The student will find it a help in following the necessary brief descriptions below to study the construction of a small dynamo, or at least careful drawings such as those in Sylvanus Thompson's 'Dynamo-Electric Machinery.'

1. *The direct current Dynamo.* In this we commonly have a single magnetic field made by powerful electro-magnets, the armature being composed of numerous turns of wire the plane of each turn passing through the axis, these are joined up to form a closed circuit. Along the spindle are placed a circle of insulated copper bars, each parallel to the axis, generally 4, 8, 16 or 32 in number, to each of which the closed conductor is attached at one point, dividing the armature symmetrically up into 4, 8, 16 or 32 coils. The current is obtained from the coils by metal brushes or current collectors which press on opposite sides of the metal strips, always in contact with the pairs of coils in which the E.M.F. is nearly or quite vanishing. Now a little consideration will show that *all* the coils as the armature rotates are sending currents in the same direction into the collectors and the external circuit connected with them. Thus the coils in each half of the armature are in series, each contributing a quota, according to its position, to the whole E.M.F., while those which form the two halves of the armature are in multiple arc. In different dynamos these coils are differently grouped, but the E.M.F. of the dynamo depends simply on the sum of the E.M.F.s of each of the coils which are *in series between the opposite brushes.*

Let the number of these be m and the angle between two successive coils β, it is obvious that everything will recur after the armature has turned through the angle β, the

angular distances of the coils from one of the brushes being given by the series

$$\theta, \quad \theta+\beta, \quad \theta+2\beta, \ldots\ldots\ldots\ldots, \theta+\overline{m-1}\beta.$$

Hence applying equation (4) the E.M.F. of the series is given by

E (for dynamo)

$$= 2 \,.\, 10^{-8}\, \pi \,.\, nSN \,(\sin\theta + \sin\overline{\theta+\beta} + \ldots + \sin\theta + \overline{m-1}\beta)$$

$$= 2 \,.\, 10^{-8} \,.\, \pi . nSN \frac{\sin\left(\theta + \dfrac{m-1}{2}\beta\right)\sin m\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}}$$

$$= 2 \,.\, 10^{-8} \,.\, \pi \,. nSN \,.\, \frac{\cos\left(\theta - \dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}}; \text{ since } m\beta = \pi.$$

This goes through all possible changes while θ changes from 0 to β.

When

$$\theta = 0, \quad E = 2 \,.\, 10^{-8}\, \pi nSN \,.\, \cot\frac{\beta}{2}.$$

$$\theta = \frac{\beta}{2}, \quad E = 2 \,.\, 10^{-8}\, \pi nSN \,.\, \operatorname{cosec}\frac{\beta}{2}.$$

$$\theta = \beta, \quad E = 2 \,.\, 10^{-8} \,.\, \pi nSN \,.\, \cot\frac{\beta}{2}.$$

Hence we see that the maximum variation of E is between values having the ratio $\cot\dfrac{\beta}{2}$ to $\operatorname{cosec}\dfrac{\beta}{2}$ or the ratio $\cos\dfrac{\beta}{2}$ to 1.

When the number of segments is great β becomes a small angle and the ratio $\cos\dfrac{\beta}{2}$ to 1 is very nearly an equality. Showing that, we obtain a nearly uniform E.M.F.

If we compute the average value of $\cos\left(\theta - \dfrac{\beta}{2}\right)$ for values

of θ between 0 and β we obtain as in the footnote the value $\frac{2}{\beta}\sin\frac{\beta}{2}$ giving

$$E \text{ (average for dynamo)} = 2 \,.\, 10^{-8} \,.\, \pi \,.\, nSN\frac{2}{\beta}$$

$$= 4 \,.\, 10^{-8}\frac{\pi}{\beta} \,.\, nSN$$

$$= 4 \,.\, 10^{-8} \,.\, mnSN \text{ volts} \,\ldots\ldots\ldots (5),$$

and if R be the resistance in ohms of the total circuit we have

$$i \text{ (average for dynamo)} = \frac{4 \,.\, 10^{-8} \,.\, mnSN}{R} \text{ amperes} \ldots\ldots (6).$$

In these formulæ it must be remembered that:

m represents the number of coils in series between the collectors and will generally be either one-half or one-quarter the number of copper sectors on the spindle, depending on the method of winding the armature:

n represents the number of revolutions per second made by the armature,

S the number of turns in each coil of the armature,

N the total magnetic flux through each loop of the armature when in the most favourable position in the field, which when the magnetizing current and dimensions and permeability of the iron in the field magnets is known, can be determined by the formula of Art. 274.

2. *The Alternate current Dynamo.* In this the planes of the coils are at right angles to the axis of the armature, and come in rapid succession as the armature rotates between pole pieces forming two crowns opposite each other near the circumference of a large iron circular plate, the successive poles on each crown being N. and S.* As the coil passes between one pair of poles lines of force are driven through it, and when it reaches the next pair lines of force in *the opposite direction* are driven through it. The series of changes in the

* In some forms of Dynamo, all the pole pieces on one crown are N. and on the opposite S., the alternating current being obtained by the magnetic flux through the armature being a maximum when between two pole pieces and a minimum when in an intermediate position.

magnetic flux through each coil is exactly similar to that undergone by the rotating coil above and the expression $SN\cos 2\pi nt$ represents well enough the law of change in the magnetic flux, provided we make $\frac{1}{n}$ sec. represent not the period of a rotation, but of a periodic change in a coil. Thus n will represent the number of complete rotations per second, multiplied by half the number of pole pieces on each crown of the field magnets.

The rate of decrease of the lines of force at a time t may also be represented as in Ex. (2) by

$$2\pi nSN\sin(2\pi nt).$$

This will not represent the E.M.F. since we must take account of self-induction.

Returning to the equation of Art. 240 we have

$$0 = Ri^2\tau + \tfrac{1}{2}L(i'^2 - i^2) + i(N' - N),$$

or dividing out by $i\tau$,

$$0 = Ri + L\frac{i' - i}{\tau} + \frac{N' - N}{\tau}.$$

But $\dfrac{N' - N}{\tau}$ = rate of increase of lines of force

$$= -2\pi nSN\sin 2\pi nt.$$

$$\therefore\quad 0 = Ri + L\frac{i' - i}{\tau} - 2\pi nSN\sin 2\pi nt \quad\ldots\ldots\ldots\ldots(6).$$

This equation cannot be solved without the use of Integral Calculus. Its solution is found to be*

$$i = \frac{2\pi SNn}{\sqrt{(2n\pi L)^2 + R^2}}\cdot\sin(2\pi nt - \phi) + C\epsilon^{-\frac{R}{L}t}\quad\ldots\ldots\ldots(7),$$

* The following proof which involves only simple differentiation may be useful.

Assume the solution to be of the form:

$$i\,.\,\epsilon^{\frac{R}{L}t} - A\epsilon^{\frac{R}{L}t}\cos 2\pi nt - B\epsilon^{\frac{R}{L}t}\sin 2\pi nt = C.$$

Differentiating with respect to t

$$\frac{di}{dt}\epsilon^{\frac{R}{L}t} + \frac{Ri}{L}\epsilon^{\frac{R}{L}t} - \frac{AR}{L}\epsilon^{\frac{R}{L}t}\cos 2\pi nt + 2\pi nA\,.\,\epsilon^{\frac{R}{L}t}\sin 2\pi nt$$
$$-\frac{BR}{L}\,.\,\epsilon^{\frac{R}{L}t}\sin 2\pi nt - 2\pi nB\epsilon^{\frac{R}{L}t}\cos 2\pi nt = 0.$$

where $\tan\phi = \frac{2\pi nL}{R}$, and C is a constant, and ϵ the base of the Napierian logarithms (= 2·7182...).

In this equation we notice that the term $C\epsilon^{-\frac{R}{L}t}$ diminishes as the time increases, and may be neglected after a short time, when the regular periodic current represented by

$$i = \frac{2\pi nSN}{\sqrt{R^2+(2\pi nL)^2}} \sin(2\pi nt - \phi) \quad \text{.........(8)},$$

is established.

And for the E.M.F. we shall have

$$E = \frac{2\pi nSNR}{\sqrt{R^2+(2\pi nL)^2}} \sin(2\pi nt - \phi) \quad \text{.........(9)},$$

In this form of dynamo there are generally as many coils as there are pole pieces so arranged that each coil is opposite a pole piece at the same time. If these coils are all in series and there are m pole pieces in the circumference the total E.M.F. is increased m times; we shall have

$$\text{(for dynamo) } E = \frac{2\pi nmSNR}{\sqrt{R^2+(2\pi nL)^2}} \cdot \sin(2\pi nt - \phi).$$

Multiply by L and divide out by $\epsilon^{\frac{R}{L}t}$,

$$L\frac{di}{dt} + Ri - (AR + 2\pi nBL)\cos 2\pi nt + (2\pi nAL - BR)\sin 2\pi nt = 0.$$

This is identical with (6) if

$$BR - 2\pi nAL = 2\pi nSN,$$
$$AR + 2\pi nBL = 0,$$

whence $$A = -\frac{4\pi^2 n^2 SNL}{R^2+(2\pi nL)^2} = -\frac{2\pi nSN\sin\phi}{\sqrt{R^2+(2\pi nL)^2}},$$

$$B = \frac{2\pi nSNR}{R^2+(2\pi nL)^2} = \frac{2\pi nSN\cos\phi}{\sqrt{R^2+(2\pi nL)^2}}.$$

where $$\tan\phi = \frac{2\pi nL}{R}.$$

Substituting the values for A, B in the given equation we have, after division by $\epsilon^{\frac{R}{L}t}$,

$$i = \frac{2\pi SN}{\sqrt{R^2+(2\pi nL)^2}} \sin(2\pi nt - \phi) + C\epsilon^{-\frac{R}{L}t}$$

agreeing with equation (7) above.

If the coils are partly in multiple arc, m will be the number in series between brush and brush.

The above formulæ are of course expressed in absolute measure, and to reduce to volts we divide by 10^8. Hence

$$\text{(for dynamo) } E = \frac{2 \cdot 10^{-8} \pi n m S N R}{\sqrt{R^2 + (4\pi n L)^2}} \sin(2\pi n t - \phi) \text{ volts} \ldots (10),$$

since E contains only the ratio of R to L it will be immaterial whether they are measured in absolute measure or R in ohms and L in secohms.

Again E goes through all its changes of value while $2\pi nt$ changes by $\frac{\pi}{2}$, and taking the average value of the sine (which will not be affected by the constant ϕ) we have for the

$$\text{(average) } E = \frac{4 \cdot 10^{-8} n m S N R}{\sqrt{R^2 + (2\pi n L)^2}} \quad \ldots\ldots\ldots (11).$$

Examining Equation (10) we notice that neglecting self-induction $L = 0$, and $\phi = 0$, and the equation reduces to

$$E = 2 \cdot 10^{-8} \pi n m S N \sin 2\pi n t.$$

Thus the effect of the self-induction on the value of E is twofold:—(i) to diminish its value in the ratio of

$$\sqrt{1 + \left(\frac{2\pi n L}{R}\right)^2} \text{ to } 1.$$

This ratio is increased both by increasing the self-induction and also by increasing the speed of rotation. Hence we see that high self-induction and high speed tend to diminish the E. M. F., and it is therefore better to keep the self-induction low by using coils without iron cores, and also to keep the speed of rotation low, securing the requisite E. M. F. by increasing N, which depends on the strength of the field magnets.

(ii) The self-induction introduces the angle ϕ, which is called the lag of the dynamo, and represents a constant time by which the periodic changes in E.M.F. and current lag behind the change in the magnetic field to which they

are due. It has, however, no influence on the average value of the E. M. F.

For the current strength at any time we have

$$i = \frac{E}{R} = \frac{2 \,.\, 10^{-8} \,.\, \pi n m S N}{\sqrt{R^2 + (2\pi n L)^2}} \sin(2\pi n t - \phi) \quad \ldots(12),$$

and the average value

$$(\text{average})\ i = \frac{4 \,.\, 10^{-8} \,.\, n m S N}{\sqrt{R^2 + (2\pi n L)^2}} \text{ amperes } \ldots\ldots(13),$$

the following being the meaning of the symbols which enter the equations.

n is the number of periodic changes per second which each coil undergoes in respect of magnetic flux through it, and is generally the number of revolutions of the armature multiplied by half the number of pole pieces on each side of the field magnet.

m is the number of coils between brush and brush and may equal the total number of coils on the armature, or this divided by 2, 4, &c. if the coils are arranged in multiple arc.

S the number of turns in each coil.

N the magnetic flux through each loop when between a pair of poles determined in accordance with Art. 274, if the magnetizing current and permeability are known.

R the total resistance in the circuit, including both armature and external circuit measured in ohms.

L the self-induction of the circuit, though generally all except that of the armature may be neglected. When all the coils are in series it will be m times the self-induction for one coil $\left(= \frac{2m\pi A S^2}{l}\right)$ where A is the area and l the thickness of the coil. Where some of the coils are in parallel the flux will be divided exactly as the current is divided between different branches, and the total flux through the coils, due to unit current between the brushes must be computed and divided by 10 to give L in secohms.

$\phi = \tan^{-1} . \frac{2\pi n L}{R}$ is the lag of the change in current behind the change in magnetic flux.

281 b. Prop. XXII. To find an expression for the output of a Dynamo.

By Art. 180 when there is a current of i amperes in circuit with an E.M.F. of E volts, the rate at which energy is being given out is measured in watts by the product Ei.

(1) In a direct current dynamo, since the variation in E and i is slight, we use the average values and get

$$Ei = \frac{1}{R}(4 \,.\, 10^{-8} mnSN)^2 \text{ watts.}$$

(2) In the alternating current dynamo we cannot use the average values. Hence

(at time t)

$$Ei = i^2 R = \frac{(2 \,.\, 10^{-8} \pi mnSN)^2 R}{R^2 + (2\pi nL)^2} \sin^2 (2\pi nt - \phi) \text{ watts.}$$

To find the average output we must take the average of all values $\sin^2 \theta$ as θ changes from 0 to $\frac{\pi}{2}$. This is easily seen to be $\frac{1}{2}$.*

Hence we have

$$(\text{Average})\, Ei = \frac{2\,(10^{-8} \pi mnSN)^2 R}{R^2 + (2\pi nL)^2} \text{ watts.}$$

281 *c*. In connection with this subject we notice the theory of transmission of energy by means of electric currents.

Looking at the expression Ei for the output of a dynamo it appears that the same energy is given out whether E is large and i small or i large and E small. Thus an output of 4000 watts may be obtained either with 4 amperes at 1000 volts or with 40 amperes at 100 volts:

Where the energy has to be transmitted to a considerable distance by metal conductors, there is an unavoidable loss of

* This may be done as for the sine by summing the series obtained on writing $\frac{1}{2}(1 - \cos 2\theta)$ for $\sin^2 \theta$ or by noticing that

$$\sin^2\left(\frac{\pi}{4} - \theta\right) + \sin^2\left(\frac{\pi}{4} + \theta\right) = 1,$$

and that for any large number of divisions the values of the squares of sines can always be put in pairs such that their sum is constant, the average is obviously half that sum or $\frac{1}{2}$.

part of the output through the heating effect on the conductor. The heat given out in a conductor is proportional to I^2R and for a given length of given material R is inversely proportional to the cross-section. Hence the heat given out is proportional to the $\frac{I^2}{S}$, where S is the cross-section, and in conductors of the same cross-section the heat generated is directly as the square of the current, or in the two cases supposed as $4^2 : 40^2$ or as $1 : 100$. To reduce the heating to the same amount we require a wire of 100 times the sectional area for a current of 40 amperes which we should require for a current of 4 amperes.

For this reason it is economical to generate electricity by a dynamo at high potential and low current, and to use some kind of transformer for transforming with as little loss as possible the high potential and low current into a low potential and high current circuit where the energy is to be consumed.

This has been secured in the case of alternate currents almost perfectly by the Alternate Current Transformer. This is an induction coil in which the primary alternating current is sent through numerous turns of wire coiled on an iron ring and generating in this ring rapidly alternating fluxes of magnetism. The secondary current is derived by coiling turns of wire as a secondary coil round the same iron ring. The magnetic circuit being of good iron and completely closed we obtain a circuit of very small reluctance in which all the lines of force generated by the primary are at each alternation forced through the coils of the secondary circuit. This change calls up an alternating induced current in the secondary.

Omitting the effects of self-induction the theory is exactly that of the Induction coil (Art. 275), the magnetic flux due to the primary being represented by $\frac{\mu A}{l} \cdot 4\pi S_1 i$ where S_1 is the number of turns in the primary and i the current strength. If S_2 be the number of turns in the secondary we

shall have for the flux through the secondary at each alternation

$$\frac{\mu A}{l} \cdot 4\pi S_1 S_2 i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and we shall have for the E.M.F. in the secondary the rate of variation (with sign changed) of this expression, which will be proportional to the rate of change in i the current strength in the primary.

The value of i for the primary is not so simple, since we have at the ends of the primary wire a periodic E.M.F. which may be represented by $E \sin 2\pi nt$.

If we make allowance for self-induction we must return to the general equation of Art. 240, which gives

$$E \sin 2\pi nt = Ri + L\frac{i' - i}{\tau}$$

which is identical in form with equation (6) and its solution may be written down by equation (8)

$$i = \frac{E \sin (2\pi nt - \phi)}{\sqrt{R^2 + (2\pi nL)^2}},$$

where $\phi = \tan^{-1} \dfrac{2\pi nL}{R}$.

E_1 the *effective* E.M.F. in the coils of the primary is now

$$E_1 = R \cdot i = \frac{ER \sin (2\pi nt - \phi)}{\sqrt{R^2 + (2\pi nL)^2}}.$$

The value of L in absolute measure is

$$L = 4\pi S_1^2 \cdot \frac{\mu A}{l},$$

being proportional to S_1^2 while R is proportional only to S_1. Hence for any kind of wire by making S_1 large enough we can make $2\pi nL$ as many multiples as we please of R, and the larger μ and n are the fewer the turns that will be required. We may therefore as an approximation neglect R^2 in the denominator in comparison with $(2\pi nL)^2$ and write the expression for the current in primary

$$i = \frac{1}{2\pi nL} \cdot E \sin (2\pi nt - \phi),$$

and for the E.M.F. effective in the coils

$$E_1 = \frac{R}{2\pi nL} . E \sin(2\pi nt - \phi).$$

Now the E.M.F. of the secondary (E_2) is measured by the rate of change in the magnetic flux through the iron core due to changes of current in the primary.

The total flux is of course Mi where M the coefficient of mutual induction

$$= 4\pi S_1 S_2 \frac{\mu A}{l}. \quad \text{(Art. 275.)}$$

The rate of increase in i is by suitable alterations in equation (2) of Art. 280

$$\frac{1}{2\pi nL} . 2\pi nE \cos(2\pi nt - \phi).$$

Hence the E.M.F. in the secondary

$$E_2 = -\frac{M}{2\pi nL} . 2\pi nE \cos(2\pi nt - \phi)$$

$$= -\frac{M}{L} . E \cos(2\pi nt - \phi)$$

$$= -\frac{S_2}{S_1} E \cos(2\pi nt - \phi).$$

Remembering that the average value of the sine or cosine between 0 and 90° is $\frac{2}{\pi}$ we have

$$\text{(average) } E_1 = \frac{2}{\pi} E,$$

$$\text{(average) } E_2 = \frac{2}{\pi} \frac{S_2}{S_1} . E,$$

$$\therefore \frac{E_2}{E_1} . = \frac{S_2}{S_1},$$

the rule made use of in practice. It is only an approximation, since we have neglected the mutual induction of the secondary on the primary in investigating the E.M.F. in the primary and the self-induction of the secondary in finding

the E.M.F. of the secondary. The first of these has the important function of making the transformer self-regulating in a remarkable degree.

The above theory in which the effect of mutual induction is omitted gives a correct result when the secondary circuit is open, or no energy is being used. Under these conditions the value of i, the current in the primary, is diminished from its value as given by Ohm's law in the ratio $2n\pi L$ to R. This means that, owing to self-induction, the current passing in the primary is very small, and the energy which would be wasted in heating conductors and making eddy currents in the iron is economised, or in other words that the Dynamo maintains the E.M.F. in the primary with a very small expenditure of energy.

Now the phases of the primary and secondary are given respectively by the $\sin 2\pi nt$ and $-\cos(2\pi nt - \phi)$ which is the same as $-\sin\left(2\pi nt + \frac{\pi}{2} - \phi\right)$ and $\tan\phi = \frac{2\pi nL}{R}$, a very large number, showing that ϕ is very nearly $\frac{\pi}{2}$.

This shows that the E.M.F. and current in primary and secondary coils are always in almost exactly opposite directions.

As soon therefore as a current passes in the secondary, the mutual induction begins to pull down the barrier offered by self-induction in the primary and causes more current to flow in the primary, which ultimately means a drain of fresh energy from the dynamo just when and only when the energy is usefully employed in the secondary. Briefly then self-induction in the primary dams back the supply of energy when it would only be wasted, while mutual induction between the secondary and primary breaks down the dam, and compels the dynamo to give, as it were on demand, a supply of energy just when it is usefully employed.

For this reason a transformer is a very efficient instrument supplying, in actual working, to the secondary circuit 96 per cent. of the energy drawn from the dynamo.

282. Prop. XXIII. To explain the action of Thomson's electric current accumulator.

This consists essentially of a circular plate revolving about an axis parallel to lines of magnetic force. The plate at one point makes contact with a fixed spring or mercury

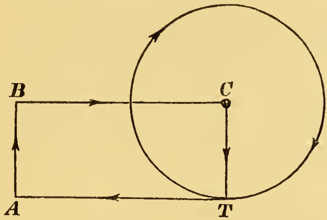

Fig. 97.

cup as T, and the circuit is completed by wires TA, AB, BC —BC being forked so as to make contact with the axis C without interfering with the rotation of the plate.

If the plate be rotated in the direction of the arrow, and the lines of force be downwards, the induced current will be in direction $CTAB$ round the closed circuit. The motion of CT clearly opposes the electromagnetic repulsion between the parallel and opposite currents CT, AB. This motion will therefore constantly tend to strengthen the induced current.

Let CT turn through an angle θ in time τ so that $\frac{\theta}{\tau} = \omega$, the angular velocity of the plate, and let a be the radius, the area traced out by the moving conductor CT is

$$\frac{1}{2} a^2 \theta.$$

The strength of the field is made up of H the magnetic strength, and $\frac{2i}{c}$ the strength of the field due to the electromagnetic action of AB, i being the current-strength, and c the distance BC (Art. 272).

Hence the electromotive force in the circuit (Art. 244, Cor. 2)

$$= \frac{1}{2} a^2 \frac{\theta}{\tau} \left(H + \frac{2i}{c}\right)$$

$$= \frac{1}{2} a^2 \omega \left(H + \frac{2i}{c}\right).$$

Then, as in Art. 240, the equation for the current will be

$$\frac{1}{2} a^2 \omega \left(H + \frac{2i}{c}\right) i\tau = Ri^2\tau + \frac{1}{2} L (i'^2 - i^2),$$

when R is the whole resistance, and i' the current-strength at the end of the small interval τ.

The case of special interest is when $H = 0$, supposing that after the current has reached a certain value i_0 the magnetic field is reduced to zero. For this case

$$\frac{a^2\omega}{c} i^2\tau = Ri^2\tau + Li(i' - i);$$

$$\therefore \left(\frac{a^2\omega}{c} - R\right)\tau = L\frac{i' - i}{i} = L \log \frac{i'}{i}$$

$$= L(\log i' - \log i).$$

If i_0 be the initial value, and i the value after a time t, we have on summation

$$\frac{a^2\omega - Rc}{c} \cdot t = L \log \frac{i_0}{i},$$

or

$$i = i_0 \epsilon^{\frac{a^2\omega - Rc}{Lc}};$$

which shows that if $\omega > \dfrac{Rc}{a^2}$, the current goes on constantly increasing in compound interest ratio.

283. Prop. XXIV. To find the value of the velocity which determines the ratio between the different electrical units in electrostatic and electromagnetic measure.

We have shown in the previous chapter that this ratio is a velocity which is independent of any system of fundamental units adopted.

Of the numerous methods which have been employed, we give two, the principles of which will be easily understood.

Method 1. To compare directly the charge of a condenser in electrostatic and electromagnetic measure.

Let a condenser be constructed of such material and form, that its capacity can easily be calculated in electrostatic measure. By means of a battery this condenser can be charged to a potential, which can be measured by an electrometer in absolute electrostatic measure. The quantity in electrostatic measure, if C represent the capacity, and V the potential, is given by

$$Q = CV \qquad (1).$$

Discharge the same condenser through a galvanometer. Then by Art. 279, if $\bar{Q}$ be its charge in electromagnetic measure,

$$\bar{Q} = \frac{2H}{\Gamma} \cdot \frac{T}{\pi} \cdot \sin\frac{\alpha}{2} \qquad (2),$$

then by Art. 255, $\dfrac{Q}{\bar{Q}} = v$, and the value of v becomes known.

284. *Method* 2. To compute the value of v in terms of a resistance.

We have shown in Chapter IX. that in electromagnetic measures resistance is of the same order as a velocity, and we defined the ohm as a velocity of 10^9 c.m. per second.

This method, due to Professor Clerk Maxwell, requires the use of a battery of very high electromotive force and a set of high resistances.

Two brass plates are placed so that one is movable, and are kept at a certain difference of potential; there is in consequence an electrostatic attraction between them. On the back of each of these plates is coiled a wire, so that the battery-current goes in the two wires in opposite directions; there will then be an electromagnetic repulsion between these currents.

The method consists in so adjusting the resistance and distance of the plates that this attraction and repulsion shall balance each other.

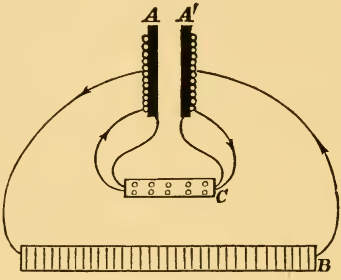

Fig. 98.

Let A, A' be the discs, B the battery, and C the large resistance. The current from the battery goes through the two coils on A, A' and through the large resistance in C. Hence if E be the difference of potential at the extremities of C in electromagnetic measure, the current-strength is given by

$$i = \frac{E}{R}.$$

Hence the repulsion between the two discs will be

$$\kappa i^2 \text{ or } \kappa \left(\frac{E}{R}\right)^2,$$

where κ depends on the geometry of the coils, and can only be computed by approximate methods.

To bring the brass plates to a difference of potential, they are connected with the terminals of C. This difference is then in electromagnetic measure E, and therefore in electrostatic measure $\frac{E}{v}$ (Art. 255).

We have shown (Art. 95) that the attraction between the plates is

$$\frac{E^2 \,.\, a^2}{8v^2b^2},$$

when a = the radius of each plate, and b = the distance between them.

Hence when the adjustments are completed,

$$\frac{E^2 . a^2}{8v^2b^2} = \kappa \frac{E^2}{R^2};$$

$$\therefore\ v = \frac{a}{2\sqrt{2\kappa} . b} R,$$

which gives v in terms of R, and R being measured in ohms can be at once converted into velocity on multiplying by 10^9.

A complete account of this method, which is due to Prof. Clerk Maxwell, together with the various adjustments required in practice, will be found in *Phil. Trans.* for 1868.

285. The results obtained by these and other methods give numerical results not very discordant, varying between 282 and 310 million metres per second, the mean being nearly 300 million metres per second. The remarkable agreement between this velocity and the various determinations of the velocity of light (which varies with different observers between 298 and 314 million metres per second), points to an intimate connection between the phenomena of electromagnetism and light. Prof. Clerk Maxwell has developed a theory of light, endeavouring to show on mechanical principles that the medium through which electromagnetic actions take place may be identical with the æther which transmits the vibrations of light.

Examples on Chapter X.

1. Show that the Moment of Inertia of a thin circular wire about an axis through its centre and perpendicular to its plane is Ma^2, where M is its mass and a its radius. Deduce the Moment of Inertia of a broad circular annulus about an axis perpendicular to its plane through its centre.

Ans. $\frac{1}{2} m (a^2 + b^2)$, where $m =$ the mass, and a, b the external and internal radii. (Cf. Chap. I. Ex. 36.)

2. Find the Moment of Inertia of a thin straight bar about an axis through one extremity.

Ans. $\frac{1}{3} ml^2$, where m is the mass and l the length.

3. Find the Moment of Inertia of the same rod about its middle point. *Ans.* $\frac{1}{12}ml^2$.

4. Show that the oscillations under gravity of a bar freely suspended by its end will be synchronous with those of a fine string two-thirds the length, having a heavy particle at its extremity.

5. A magnet A is placed so that its axis produced, bisects at right angles the axis of another magnet B, the distance between their centres being great compared to their lengths. Make an approximation to the couple produced by A on B, and that produced by B and A.

Ans. $\dfrac{8aa'mm'}{c^3}$, $\dfrac{4aa'mm'}{c^3}$, where $2a$, $2a'$ are the lengths of A and B, c the distance between their centres, and mm' their magnetisms.

6. A long magnet acts on a small compass-needle placed on its axis. Find the error produced by it on the compass in different directions of the disturbing magnet.

7. A small compass is placed at the centre of a long magnet whose axis makes an angle θ with the magnetic meridian. If δ be the deviation of the compass-needle $\cot\delta \pm \cot\theta = \dfrac{H}{F}\operatorname{cosec}\theta$, where H, F are the strengths of the earth's field and the magnet's field at the compass.

8. One end of a magnet is prolonged by a thin stem of gumlac which carries a small pith-ball, the other end having a counterpoise. An equal ball is so fixed that the two are just in contact when the magnet is in the meridian. The two balls are electrified to a potential V, and the magnet is observed to be deflected through an angle 2α; show that V^2 varies nearly as $(\sin\alpha)^3$.

9. A hole is pierced in a card through which passes a straight wire carrying a current. Iron filings are sprinkled over the card, and the card gently tapped. Find the form assumed by the iron filings.

10. If a magnet be placed anywhere in the magnetic field due to a straight current, show that the magnet has no tendency to rotate, as a whole, round the current.

11. Deduce the principles which guide us in experiments on the rotation of a magnet round a current, and a current round a magnet.

12. In the experiment of the last question, show that the whole amount of work spent in each rotation of the magnet pole round the current or vice versa is $4\pi mi$, where m is the strength of the pole and i the current-strength in the conductor, whose length is supposed to be infinite.

13. A magnet is suspended in a horizontal plane so as to be free to move about its south pole, and a vertical current is approached towards it.

(i) The conductor being outside the circle described by the north pole, show that the north pole will be driven by the current to rotate in opposite directions through portions of the circumference bounded by tangents to the circle from the intersection of the plane of the circle by the conductor.

(ii) The conductor being within the same circle, the direction of movement of the north pole will be in all parts of the circumference the same.

(iii) The conductor being on the circumference of the circle, show that the rotation will be always in the same direction.

(iv) Show that no permanent rotation of the magnet can be produced by this means.

14. A magnet NS is supported at its middle point, and a conductor carrying a downward current cuts the horizontal plane at O.

(i) A circle is drawn about the triangle ONS, and a diameter drawn through O. From N, S perpendiculars Na, Sb are drawn on to this diameter. Show that in all positions the moment of the forces on the magnet turning its north pole in a direction right-handed to the conductor is

$$\frac{Gi}{ON \cdot OS}(Sb \pm Na),$$

G being the moment of the magnet, and the plus sign being employed when the perpendiculars fall on the same side of the diameter.

(ii) If the conductor cut the circumference of the circle of which the magnet is a diameter, there is no tendency to rotate the magnet.

(iii) If the conductor be outside the circle, the direction of rotation is governed by that of the more remote pole.

(iv) If the conductor be within the circle, the direction of rotation is governed by that of the nearer pole.

(v) If the conductor be placed on the line bisecting the magnet at right angles, the rotative force will be nil.

(vi) If the field be divided by the circle of which the magnet is the diameter, by the magnetic axis produced, and by a line bisecting it at right angles; show that on crossing any of these lines if on one side the current appear to attract the north pole of the magnet, on the opposite side it appears to repel it.

(vii) Draw a diagram showing in what positions the conductor appears to attract the magnet, and in what positions it appears to repel it.

15. Show that in measuring a current by a sine galvanometer if the current be stronger than a certain limit, it will be necessary to shunt the current before measuring it.

16. If a tangent galvanometer be arranged so that it can also be used as a sine galvanometer, show that any current producing more than 45° deflection in the instrument, when used as a tangent galvanometer, must be shunted before being measured by it as a sine galvanometer.

17. In Helmholtz's arrangement for a tangent galvanometer, show that the greatest degree of constancy of magnetic field along the axis near the magnet will be when the distance between the coils is equal to the radius of either coil.

18. Show that in the galvanometer of the last question the galvanometer-constant is given by $\Gamma = \dfrac{32\pi}{4\sqrt{5}\,.\,a}$, where a is the radius of the coil.

19. A finite wire carrying a current is perpendicular to and on one side of an infinite wire also carrying a current. Find the magnitude and direction of the force exerted by the latter upon the former wire.

Ans. $2ii' \log \frac{y_1}{y_2}$, where i, i' are the current-strengths, and y_1, y_2 the distances of the ends of the finite from the infinite wire. The direction will be parallel to the current in the infinite wire when the current in the perpendicular wire is away from it.

20. If the length of a helix be forty times its diameter, show that the strength of the magnetic field within it varies about one-thousandth part through $\frac{2}{3}$ of its length.

21. A helix A is placed with its axis perpendicular to the meridian, and a short magnet B is suspended at a point on its axis produced, the magnet being deflected from the meridian by a current in the helix. Another magnet C is now placed with its axis along the axis of the helix produced and moved about till B is again in the meridian.

If $2l'$ be the length, and G the moment of C, $2l$ the length, and Ai the moment of A (i being current-strength), a' and a the distances of the middle points of A and C from the suspension of B, then show that

$$\frac{G}{l'}\left\{\frac{1}{(a'-l')^2}-\frac{1}{(a'+l')^2}\right\}=\frac{Ai}{l}\left\{\frac{1}{(a-l)^2}-\frac{1}{(a+l)^2}\right\}.$$

22. A current is generated in a circuit and the electromotive force suddenly removed, find the law of decrease of current.

Ans. If i_0 be current at first that after a time t is

$$i_0\epsilon^{-\frac{R}{2L}\cdot t}.$$

23. A small sphere of soft iron is suspended at one end of a lath, which is counterpoised and delicately suspended at a point near the other end, so that the sphere moves on the arc of a large circle which may be considered approxi-

mately a straight line. Two opposite magnet poles of different strengths are placed at different points on the same side of the sphere in its line of motion. Find the positions of unstable and stable equilibrium.

Ans. Unstable when the distances from the poles are in the ratio of the square roots, but stable when in the ratio of the cube roots of the strengths of the poles.

24. The same sphere moves along the line which bisects at right angles the distance between two equal and similar magnet poles. Show that there is a point of stable equilibrium at a distance $\frac{a}{2\sqrt{2}}$ on either side of the line joining the poles, a being the length of the line.

25. The strength of the magnetic field at any point within a plane circular current whose strength is unity is given by the perimeter of the ellipse concentric with the circle, and which has the given point for focus, divided by the square on the semi-minor axis.

26. If the point be outside the circular current the strength will be the defect from its asymptotes of the concentric hyperbola, which has the given point for focus, divided by the square on the semi-conjugate axis.

27. Find the Horse-Power required to maintain a current of ·75 ampere in each of 100 lamps of 45 volts arranged (1) each lamp connected in parallel with the dynamo by a lead of 1 ohm resistance, (2) in 10 rows, each of 10 lamps, the leads to each row having a resistance of 10 ohms.

Ans. 4·6 in both cases.

28. Two horizontal rods are placed parallel at a distance of 1 metre, and a third rod is placed across them and slides parallel to itself at a rate of 10 metres per second. Find in volts the E.M.F. between the ends of the fixed rods, assuming the earth's vertical magnetic force to be ·47 C.G.S. units.

Ans. ·00047 volts.

CHAPTER XI.

THERMO-ELECTRICITY.

286. In the cases we have hitherto considered the energy of a voltaic current is derived either from chemical action (as in a battery) or Mechanical work (as in a dynamo engine). In the currents we have now to consider the energy is derived from the *unequal* heating of the different parts in a compound circuit, the passage of the current causing an absorption of heat at some parts of the circuit in excess of that evolved at other parts.

The laws regulating the development of these currents have been discovered by a series of experiments, the results of which we now proceed to state briefly.

287. Experiment 1. *Seebeck discovered that if bars of two metals (bismuth and antimony) were soldered at their ends and the junctions brought to different temperatures an electric current flowed round the circuit; flowing through the hot junction from bismuth to antimony.*

Seebeck concluded that the electromotive force of this current was proportional to the difference of temperature at the junctions, a result only true for small ranges of temperature, unless the mean temperature be kept constant.

288. Experiment 2. *Peltier discovered that if a current (from a battery or dynamo-engine for instance) be sent through an arc of several metals heat is absorbed at some junctions and emitted at others; the emission and absorption being exactly reversed by reversing the direction of the current; the quantities of these thermal actions being proportional to the current strength.*

This observation has led in the hands chiefly of Thomson and Tait to a theory of Thermo-electricity founded on the laws of Thermo-dynamics.

Suppose an arc of different metals to have its terminals of the same metal at the same temperature and suppose between these terminals a constant Electromotive Force F is established, causing a current of strength i to pass round the circuit. The energy per unit time sent into the circuit is therefore Fi. This is partly used up in frictional generation of heat whose amount by Ohm's law is Ri^2 and partly in heat absorbed or evolved according to Peltier's law (though not entirely due, as we shall see, to the Peltier effect). Let the total amount of heat absorbed per unit current per unit time be denoted by A, that actually absorbed will be Ai per unit time.

Hence $$Fi = -JAi + Ri^2,$$

J denoting, as usual, the mechanical equivalent of heat,

$$\therefore \; i = \frac{F + JA}{R}.$$

The form of this expression shows that the *effective* E.M.F. of the circuit is $F + JA$. If this vanish the impressed E.M.F. just balances the Thermo-electro-motive Force: or $F = -JA$.

Hence if E be the E.M.F. of any Thermo-electric arrangement, and ΣH be the sum of all the heat absorbed or evolved per unit current per unit time, *according to Peltier's law*, and counted positive when evolved,

$$E + J\Sigma H = 0 \qquad (1).$$

This equation is the application of the first law of Thermo-dynamics to Thermo-electricity. Since the quantities of heat in equation (1) are all reversed in sign by the reversal of the current; if this were all the heat developed in the circuit, it would obey Carnot's law of reversibility, on which the application of the second law of Thermo-dynamics depends. If the section of the conductor be large and the current i be small enough the term Ri^2 depending on the square of i may be neglected in comparison with $-Ai$ which

depends on i. On this supposition we may apply to the system the second law of Thermo-dynamics, which leads to the equation

$$\Sigma \frac{H}{T} = 0 \quad \text{.......................} \quad (2),$$

where the elements of $\Sigma \frac{H}{T}$ represent the quotient of each quantity of heat in (1) divided by the *absolute* temperature at which it is evolved or absorbed.

289. EXPERIMENT 3. *It was shown by the late Prof. J. Cumming that for copper and iron there was a certain temperature* (*about* 284° C.) *at which the Peltier effect vanished, so that the metals are at that temperature neutral to each other, and if one junction be kept at this neutral temperature the current is in the same direction whether the second junction be at a higher or lower temperature.*

From this observation Thomson argues thus.—Since in every thermal engine the energy is derived from an absorption of heat at the hotter part in excess over that evolved at the colder, the energy of the Thermo-electric current, so far as it depends on the Peltier effect must be due to an excess of heat absorbed at the hot junction over that evolved at the cold junction. If therefore the hotter junction in an Iron-Copper couple be kept at the neutral temperature, the Peltier effect must give an evolution of heat at the colder junction, and we should have a Thermo-electric current without any absorption of heat: unless there be in the Iron-Copper circuit an absorption of heat, distinct from the Peltier effect at the junctions; though like the Peltier effect reversible with the current. This can only consist in an absorption of heat when the current passes from a hotter to a colder part or *vice versâ* in one or both metals. Thomson has shown by numerous experiments that in an unequally heated copper conductor electricity behaves analogously to a real fluid, tending to reduce the differences in temperature, while in an iron conductor it tends to exaggerate them. This "Thomson effect" is sometimes known as *Electric Convection of Heat.*

290. Experiment 4. *Sir W. Thomson showed that generally in an unequally heated body there is an absorption or evolution of heat owing to the passage of a current from cold to hot or from hot to cold; this effect, like Peltier's, being reversible with the current.*

Experiment 5. *Magnus showed that no current can be produced by any variations of temperature in a circuit of a single homogeneous conductor.*

These experiments give us means of measuring the quantities of heat (H) in equations (1) and (2) which must include those due both to the Peltier and Thomson effect.

291. To measure the Thomson effect we first observe that in a homogeneous circuit, if Θ_{xy} denote the quantity of heat absorbed per unit time by unit current going from a place of higher absolute temperature x to one at lower absolute temperature y in direction from x to y; Θ_{yz}, Θ_{zx}, similar things for other parts of the circuit supposed to consist of only one metal, the whole heat absorbed by unit current in the complete circuit

$$= \Theta_{xy} + \Theta_{yz} + \Theta_{zx},$$

and this will equal the whole electromotive force in the circuit which by Magnus' result must vanish,

$$\therefore \; \Theta_{xy} = \Theta_{xz} - \Theta_{yz},$$

showing since z is arbitrary that the Thomson effect in any homogeneous arc is represented by the difference of some function of the temperature at one end and the same function of the temperature at the other end. We may write this

$$\Theta_{xy} = \phi(x) - \phi(y).$$

It was suggested by Prof. Tait and confirmed by experiment, in the case of most metals within the ordinary range of temperature, that the form of $\phi(x)$ might be for any one metal $\frac{1}{2}k_a x^2$, where k_a is a constant depending on the particular kind of metal; positive for metals like copper and negative for metals like iron. Thus

$$\Theta_{xy} = \tfrac{1}{2}k_a(x^2 - y^2) \qquad \text{.....................(2)}$$

$$= k_a(x - y)\left(\frac{x+y}{2}\right).$$

If $x-y$ be small and $\dfrac{x+y}{2}=T$, the mean temperature, we have for the Thomson effect per degree of temperature at mean temperature T, $k_a T$.

In equation (1) the terms due to the Thomson effect will be of the form Θ_{xy}. In equation (2) we must compute the value of $\dfrac{H}{T}$ for the heat evolved from each element of the circuit. Let then H be the heat evolved from an element whose terminals differ in temperature by a very small quantity τ. By the last equation $H=k_a T\tau$, where T is the absolute temperature of the element. Hence each term in $\Sigma\dfrac{H}{T}$ will be of the form $k_a\tau$, and the whole term corresponding to the metal a will be

$$\Sigma k_a\tau = k_a(x-y)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and similar terms for the other metals.

The Thomson effect per degree of temperature is often called the *Specific Heat of Electricity*, and it is found by multiplying the coefficient k_a for the metal, by the absolute temperature.

292. To measure the Peltier effect we will denote by Π_t the quantity of heat evolved in unit time by the passage of unit current across a junction of two metals at the absolute temperature t.

293. Our equations (1) and (2) derived from the first and second laws of Thermo-dynamics may now be written respectively;

$$E+J\{\Sigma\,.\,\Pi_t-\tfrac{1}{2}\Sigma k(x^2-y^2)\}=0\ \ldots\ldots\ldots(A).$$

$$\Sigma\frac{\Pi_t}{t}-\Sigma k(x-y)=0\ \ldots\ldots\ldots(B).$$

294. Prop. I. If any circuit of different metals be throughout at the same temperature the sum of the Peltier effects at the junctions vanishes.

In this case $x = y = t$ in all terms of (A) and (B). Hence in (B)

$$\Sigma \frac{\Pi_t}{t} = 0, \text{ or } \frac{1}{t} \Sigma \Pi_t = 0,$$

or the sum of Peltier effects vanishes.

295. **Prop. II. To express Π_x as a function of x and constants.**

Let the circuit consist of two metals a, b, and let the temperatures of the junctions be x, y, and suppose the current to pass from a to b through the junction at temperature x. Supposing the Peltier effect positive at this junction it will be negative at the opposite.

Hence equation (B) gives

$$\frac{\Pi_x}{x} - \frac{\Pi_y}{y} = k_a(x - y) + k_b(y - x)$$
$$= (k_a - k_b)(x - y);$$

$$\therefore \quad \frac{\Pi_x}{x} - (k_a - k_b)x = \frac{\Pi_y}{y} - (k_a - k_b)y.$$

This shows that for all values of x

$$\frac{\Pi_x}{x} - (k_a - k_b)x = \text{a constant} = C, \text{ suppose.}$$

Let T_{ab} be the neutral temperature, at which the Peltier effect vanishes; so that $\Pi_x = 0$ when $x = T_{ab}$.

$$\therefore \quad -(k_a - k_b)T_{ab} = C,$$

$$\therefore \quad \frac{\Pi_x}{x} = (k_a - k_b)(x - T_{ab}),$$

$$\therefore \quad \Pi_x = (k_a - k_b)(x - T_{ab})x \text{ ....(4).}$$

296. **Prop. III. To prove that for any three metals a, b, c**

$$(k_a - k_b)T_{ab} + (k_b - k_c)T_{bc} + (k_c - k_a)T_{ac} = 0.$$

For distinction, let us suppose the Peltier effect at the three junctions, all at the same given temperature, to be denoted by Π_{ab}, Π_{bc}, Π_{ca} respectively.

We have shown Prop. (i) that when temperature is constant

$$\Pi_{ab} + \Pi_{bc} + \Pi_{ca} = 0.$$

Substitute for Π_{ab}, Π_{bc}, Π_{ca} their values from (4) and dividing through by x

$$(k_a - k_b)(x - T_{ab}) + (k_b - k_c)(x - T_{bc}) + (k_c - k_a)(x - T_{ac}) = 0.$$

$$\therefore \quad (k_a - k_b) T_{ab} + (k_b - k_c) T_{bc} + (k_c - k_a) T_{ac} = 0.$$

297. Prop. IV. To express the E.M.F. for a circuit of two metals with junctions at given temperatures in terms of the temperatures and constants.

Let as before the temperatures of the junctions be x and y.

Equation (A) gives us

$$-\frac{E}{J} = \Pi_x - \Pi_y - \tfrac{1}{2}(k_a - k_b)(x^2 - y^2)$$

$$= (k_a - k_b)\{x^2 - y^2 - Tab(x - y)\} - \tfrac{1}{2}(k_a - k_b)(x^2 - y^2)$$

$$= (k_a - k_b)(x - y)\left(\frac{x + y}{2} - Tab\right) \quad \text{.................. (5)},$$

which gives the E.M.F. required.

Cor. Suppose $x - y = \tau$, a very small quantity, and

$$\frac{x + y}{2} = T, \text{ the mean temperature;}$$

then we have

$$\frac{E}{\tau} = J(k_a - k_b)(T_{ab} - T) \quad \text{................(6)}.$$

The ratio $\frac{E}{\tau}$ is the electromotive force per degree of difference of temperature between the hot and cold junctions, which have the mean temperature T. This may be called the thermo-electric power of the pair a, b at temperature T.

Def. *The Thermo-electric power of a given pair of metals at given mean temperature is the electromotive force of the Thermo-electric circuit per degree difference in temperature of the junctions.*

298. Prop. V. The sum of the thermo-electric powers at the same temperature of three metals taken two and two vanishes.

Let us denote the thermo-electric power of the pairs of metals at given temperature by $E(a, b)$, $E(b, c)$, $E(c, a)$.

Then
$$E(a, b) = J(k_a - k_b)(T_{ab} - T)$$
$$E(b, c) = J(k_b - k_c)(T_{bc} - T)$$
$$E(c, a) = J(k_c - k_a)(T_{ac} - T),$$

on addition, and by help of Prop. III. we see that

$$E(a, b) + E(b, c) + E(c, a) = 0.$$

Cor. This result may be written

$$E(a, b) = E(a, c) - E(b, c) \quad \text{............} \quad (7),$$

which shows that the thermo-electric power of two metals is the difference of their separate thermo-electric powers referred to any third metal.

299. We will now explain a graphical method of indicating the thermo-electric properties of a circuit first suggested by Sir W. Thomson and developed by Prof. Tait.

Let E_t denote the thermo-electric power of two given metals at temperature t. Then equation (6) may be written

$$\frac{E_t}{J} = k_a(T_{ab} - t) - k_b(T_{ab} - t).$$

Let now
$$\left.\begin{aligned} z &= k_a(T_{ab} - t) \\ z' &= k_b(T_{ab} - t) \end{aligned}\right\} \quad \text{.....................} \quad (8),$$

whence
$$\frac{E_t}{J} = z - z'.$$

If we represent by abscissæ temperatures counting from absolute zero, and by ordinates the values of z and z' in (8), each equation represents a straight line, and the thermo-electric power bears a constant ratio to (and may therefore

be measured by) the difference of the ordinates of two given straight lines, corresponding to the same abscissa.

We may still further simplify the construction if we can find a metal in which the Thomson effect is nil. Le Roux's observations tend to show that this is the case in lead. In this case if b represent lead $k_b = 0$, and therefore $z' = 0$, and the thermo-electric power of every metal referred to lead is given by

$$-\frac{E_t}{J} = k_a (t - T_{ab}) \quad \text{.................} (9).$$

If then we call the line of abscissæ the lead line we can, after determining from observation the values of k_a and T_{ab} for each metal, construct the line represented by this equation for that metal and thus find graphically the Thermo-electric Power of every metal referred to lead. And by applying Prop. V. Cor. we see that the thermo-electric power of every other pair of metals can be at once obtained from the diagram by measuring the difference of the ordinates for those metals corresponding to a given abscissa or temperature.

300. Let us construct the diagram for two metals, the line AB representing copper in which k, the coefficient of the Thomson effect, is positive, and CD representing iron in which k is negative, and the line is therefore inclined in the opposite direction. These will intersect in the neutral point corresponding to temperature 284° C. In drawing the figure $-\frac{E_t}{J}$ is made the ordinate, which therefore represents Thermo-electric power with sign reversed. Thus lead is positive to any metal where its ordinate is above the line of abscissæ.

Let us consider a circuit made by a junction BD having a temperature OQ, and another junction AC, having a lower temperature OP.

We can now give a geometrical interpretation to the equations we have proved above.

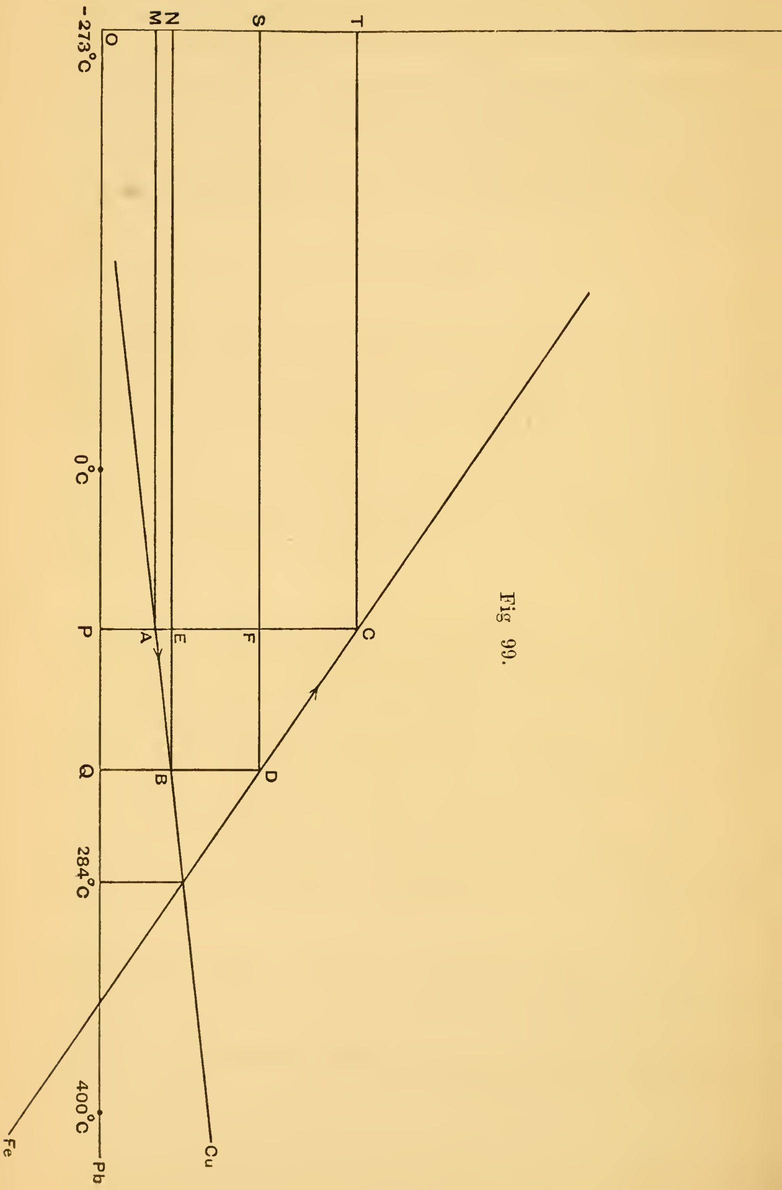

Fig 99.

(i) Denoting iron by a and copper by b in the above notation, and the temperatures OP, OQ by x, y respectively, we have

$$DQ = k_a (T_{ac} - x) \text{ and } BQ = k_b (T_{bc} - x),$$

$$CP = k_a (T_{ac} - y) \text{ and } AP = k_b (T_{bc} - y).$$

Equation (8) shows that the thermo-electric power at temperature OQ is BD and at temperature OP is AC.

(ii) By equation (2) the Thomson effect in the copper is given by

$$\begin{aligned}\Theta_{xy} &= k_a (x - y) \left(\frac{x + y}{2}\right)\\ &= \{k_a (T_{ab} - y) - k_a (T_{ab} - x)\} \frac{x + y}{2}\\ &= (CP - DQ) \,.\, \frac{OP + OQ}{2}\\ &= CF \,.\, \frac{OP + OQ}{2} = ST \,.\, \frac{OP + OQ}{2}\\ &= \text{area } CDST.\end{aligned}$$

By similar reasoning the Thomson effect in AB is given by $-$area $ABNM$, the minus sign arising from the factor $(AP - BQ)$.

(iii) By equation (4) the Peltier effect at temperature x is given by

$$\begin{aligned}\Pi_x &= (k_a - k_b)(x - T_{ab})\, x\\ &= x \{k_b (T_{ab} - x) - k_a (T_{ab} - x)\}\\ &= OQ \{BQ - DQ\}\\ &= - OQ \,.\, BD\\ &= - \text{area } BDSN.\end{aligned}$$

Similarly the Peltier effect at the junction AC will be given by $+$area $ACTM$.

(iv) For the E.M.F. of the circuit we have by equation (5)

$$\frac{E}{J} = (k_a - k_b)(x - y)\left(T_{ab} - \frac{x + y}{2}\right)$$
$$= \tfrac{1}{2}(k_a - k_b)(x - y)(T_{ab} - x + T_{ab} - y)$$
$$= \tfrac{1}{2}(x - y)\{(k_a - k_b)(T_{ab} - x) + (k_a - k_b)(T_{ab} - y)\}$$
$$= \tfrac{1}{2} PQ \,.\, (BD + AC)$$
$$= \text{area } ABDC.$$

(v) For the direction of the current we need only notice that it must cause an *absorption* of heat at the hot junction. Referring to Art. 295 we see that Π_x is defined in the type case as "the quantity of heat evolved by unit current in passing from a to b." The above investigation shows that Π_x is positive at the cold junction and therefore the current passes from iron to copper through the cold junction and from copper to iron through the hot junction. It is in fact in the direction of the arrows, circulating round $ABDC$ in the positive direction of angular measurement.

301. Before leaving the diagram it should be noticed that the original assumption of Art. 291 that k is a constant at all temperatures for the same metal has never been demonstrated by experiment. Should it ever be proved that k is a function of the temperature equations (8) will no longer represent straight lines. It is however certain from experiment within ordinary temperatures that $(k_a - k_b)$ the coefficient for the E.M.F. of a circuit formed of two metals (Art. 297) is independent of the temperature. Thus our straight lined Thermo-electric diagram could be converted into the true diagram by a simple shear parallel to the ordinates.

302. These remarks must be strictly confined to ordinary temperature, or about from $-18°$ C. to $350°$ C. Prof. Tait experimenting at higher temperature has shown in the case of at least two metals, iron and nickel, remarkable aberrations. With reference to iron he says (Rede Lecture, 1873, Report in *Nature*, Vol. VIII. p. 122), "The cause (of the irregularity in iron) is this, that while, as Thomson discovered,

the specific heat of [electricity in] iron is negative at ordinary temperatures, it becomes positive at some temperature near low red heat; and remains positive till near the melting point of iron, when it appears possible from some of my experiments that it may again change sign." Thus the line for iron in the diagram at a high temperature bends upwards, and possibly at a still higher temperature, but before its melting point, bends downwards again. It thus appears on the diagram that iron becomes neutral to copper and to lead each at two different temperatures, and possibly to a compound of platinum and iridium at three different temperatures.

EXAMPLES ON CHAPTER XI.

1. The neutral temperatures with lead of zinc and iron are respectively $-95°$ C. and $+356°$ C., while the coefficients of specific heat (k) for the same two metals are respectively $+\cdot00122$ and $-\cdot00247$. Calculate the neutral point of zinc and iron. *Ans.* 207° C.

2. At 20° C. the thermo-electric powers relatively to lead expressed in microvolts are found to be for copper 1·5, and for iron 17·5, while their neutral temperatures relatively to lead are $-132°$ C. and $+356°$ C. Calculate the coefficients of specific heat on this scale.

Ans. For copper ·0098, and for iron $-\cdot052$.

3. From the same data as question 2 calculate in microvolts the E.M.F. of an iron-copper pair whose junctions are respectively at 0° C. and 100° C.

Ans. 1489 microvolts.

4. Show that in an iron-copper pair if the cold junction be kept at a fixed low temperature and the other junction be heated the current will gradually rise, reach a maximum and then gradually sink again. What is the temperature of the maximum?

5. Three wires A, B, C of different metals and resistances a, b, c are soldered together at two junctions which are maintained at two different temperatures. If I_a be the current when A is cut and I_b the current when B is cut, show that the current in C when all the wires are continuous will be

$$\frac{a(b+c)I_a + b(a+c)I_b}{ab+bc+ca}.$$

6. If the arc C contains a galvanometer of large resistance compared to a and b, show that the current in C is

$$\frac{aI_a + bI_b}{a+b}.$$

7. If E, E' be the thermo-electric powers of the wires A, B in question 6, relatively to lead, the thermo-electric power of the compound wire A, B will be

$$\frac{aE' + bE}{a+b}.$$

8. Hence show that the line for the compound wire A, B in the thermo-electric diagram passes through the neutral point of A, B and by properly adjusting the ratio $a:b$ can be made to take any required position between the lines for A and B.

9. A couple is made of platinum-iridium (in which the Thomson effect vanishes) and iron. Each junction is kept at one of the neutral temperatures of platinum-iridium and iron. Discuss fully the thermal conditions of the circuit.

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